

**Shilnikov Chaos, Low Interest Rates, and New Keynesian
Macroeconomics**

William A. Barnett, Giovanni Bella, Taniya Ghosh, Paolo Mattana and Beatrice Venturi



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Abstract

The paper shows that in a New Keynesian (NK) model, an active interest rate feedback monetary policy, when combined with a Ricardian passive fiscal policy, à la Leeper-Woodford, may induce the onset of a Shilnikov chaotic attractor in the region of the parameter space where uniqueness of the equilibrium prevails locally. Implications, ranging from long-term unpredictability to global indeterminacy, are discussed in the paper. We find that throughout the attractor, the economy lingers in particular regions, within which the emerging aperiodic dynamics tend to evolve for a long time around lower-than-targeted inflation and nominal interest rates. This can be interpreted as a liquidity trap phenomenon, produced by the existence of a chaotic attractor, and not by the influence of an unintended steady state or the Central Bank's intentional choice of a steady state nominal interest rate at its lower bound. In addition, our finding of Shilnikov chaos can provide an alternative explanation for the controversial “loanable funds” over-saving theory, which seeks to explain why interest rates and, to a lesser extent inflation rates, have declined to current low levels, such that the real rate of interest is below the marginal product of capital. Paradoxically, an active interest rate feedback policy can cause nominal interest rates, inflation rates, and real interest rates unintentionally to drift downwards within a Shilnikov attractor set. Policy options to eliminate or control the chaotic dynamics are developed.

Keywords: Shilnikov chaos criterion, global indeterminacy, long-term un-predictability, liquidity trap.

JEL Code: C61, C62, E12, E52, E63

*Shilnikov Chaos, Low Interest Rates, and New Keynesian Macroeconomics**

William A. Barnett¹, Giovanni Bella², Taniya Ghosh³, Paolo Mattana⁴, and Beatrice Venturi⁵

January 30, 2020

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1. Introduction

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1.1. Prior research on chaos in economics

A long literature exists on the search for policy relevant chaos in economics. The earliest literature used tests developed by physicists for detecting chaos in data produced from controlled experiments. Those tests focused primarily on measuring the Hausdorff dimension of the attractor set and testing for positive dominant Liapunov exponent.⁶ Using those tests, Barnett and Chen (1988a,b) found chaos in monetary aggregate data. Many relevant published papers followed. But since the data were not produced from a controlled experiment and the tests did not condition on an economic model, the tests had no way to impute the source of the chaos to the nonlinear dynamics of the economy. For example, the source of the chaos could be from the weather or the climate impacting the economy.

Attention then moved to the possibility of formal statistical testing of the null hypothesis of chaos within a dynamic macroeconomic model. But this approach was found to be prohibitively difficult. Analytical solution for the boundaries of the chaotic subset of the parameter space is not currently possible with more than three parameters. Iterative numerical search for that subset is possible. But even if the set has been found, the geometric properties of the set, which may be a fractal, violate the regularity conditions for available statistical sets. For example, the likelihood function can have singularities as it passes over the null hypothesis set.⁷

Faced with such statistical inference problems, research turned to exploration of the theoretical properties of macroeconomic models. In a famous paper, Grandmont (1985) found that a classical model's parameter space is stratified into an infinite number of subsets separated by period doubling bifurcation boundaries. Based upon the subset within which the parameters are located, the solution of the model could be monotonically stable, damped stable, periodic unstable, multiperiodic, or – after a converged infinite number of bifurcations – chaotic. But since the classical model contains no market imperfections, all solutions are Pareto Optimal. Hence no clear reason exists for governmental intervention. If the parameters are in the chaotic region, then the chaos is Pareto Optimal, and governmental attempts to control the chaos could produce a Pareto loss, harming welfare. In addition, it has been speculated that the parameter settings leading to chaos in a classical model may not be plausible (see, e.g., Blanchard and Fischer (1989, p. 261)).

Research then turned to exploring the theoretical properties of NK models, within which governmental intervention can be justified. Findings of chaos in a NK model at plausible settings of parameters, when policy is based on active interest rate feedback rules, such as the Taylor Rule, have been reported by Benhabib et al (2002). But implications of that chaos for policy are not clear, with available fiscal policy options appearing to be ineffective as possible solutions to the problem.

Policy relevance of chaos in economics depends not only upon the existence of chaos at plausible settings of parameter values, but also the nature of the chaos. Many kinds of chaos exist. For example, Li-Yorke chaos (for dynamical systems generated by interval maps), Lorentz attractor (a type of chaos from atmospheric dynamical model), Smale horse-shoe chaos (with origin from celestial mechanics),

⁶ For a relevant newer approach, see Kuznetsov (2016).

⁷ See, e.g., Geweke (1992). Inference procedures that might be applicable under such nonstandard conditions are extremely difficult to apply and have not been attempted. See, e.g., Section 4 of Geweke and Durham (2019).

and Shilnikov chaos for a particular scenario/criterion related to the Shilnikov homoclinic orbit. We chose to investigate Shilnikov chaos for multiple reasons. One reason is that it can be detected directly from the Shilnikov criterion. But also, as explained by Alan Champneys (2011), “Over the years, Shilnikov’s mechanism of chaos has proven to be one of the most robust and frequently occurring mechanisms chosen by nature.”

As pointed out by Afraimovich et al (2014, p. 19), “Only starting from mid 70s–80s, when researchers became interested in computer studies of chaotic behavior in nonlinear models, it became clear that the Shilnikov saddle-focus is a pivotal element of chaotic dynamics in a broad range of real-world applications. In general, the number of various models from hydrodynamics, optics, chemical kinetics, biology etc., which demonstrated the numerically or experimentally strange attractors with the characteristic spiral structure suggesting the occurrence of a saddle focus homoclinic loop, was overwhelming. Indeed, this scenario has turned out to be typical for a variety of systems and models of very diverse origins.” The relevancy of this general observation to economics has been confirmed by the finding of Shilnikov chaos in an economic growth model by Bella, Mattana, Venturi (2017). We find potentially high relevance of Shilnikov chaos to current problems in the world’s macroeconomies, when active Taylor rule monetary feedback policy is adjoined to a NK dynamic macroeconomic model.

1.2. Our approach

As stated by Christiano and Takahashi (2018), “Monetary models are notorious for having multiple equilibria. The standard NK model, which assumes that fiscal policy is passive and monetary policy is set by a Taylor rule is no exception.” In fact, a large literature exists on complicated dynamics problems produced by NK models with Taylor rule interest rate feedback policies. Our research introduces new problems, which we believe are potentially highly relevant to policy challenges in recent years. We also propose potential solutions to the problems.

Many papers have shown that following an aggressive interest rate policy, in accordance with the Taylor Principle, is not a sufficient criterion for stability in the NK model.⁸ One major obstacle to uniqueness is that the stance of fiscal policy may collide with the central bank’s inflation objective, when fiscal policy is unable or unwilling to adjust primary surpluses to stabilize government debt (Kumhof *et al.*, 2010).⁹ Further limits may result from the way preferences and technologies are introduced into the model.

⁸ For example, following the Taylor Principle is not sufficient in the presence of nominal capital income taxation (Røisland (2003) and Edge and Rudd (2007)), or in the presence of high government consumption (Natvik (2009), Galí et al. (2004)), or in the presence of trend inflation (Coibion and Gorodnichenko (2011) and Kiley (2007)). The greater the effective capital income tax or the greater the government consumption, the more aggressively the interest rate should respond to inflation, to attain a determinate equilibrium.

⁹ Regardless of the stance of fiscal policy, the role that the demand for money by agents plays in the monetary-transmission mechanism may undermine uniqueness of the equilibrium and stimulate the onset of expectation-driven fluctuations (cf. Benhabib *et al.* 2001a,b). Sveen and Weinke (2005, 2007) show that inclusion of firm-specific capital in a standard NK model can lead to multiple equilibria with aggressive interest rate policies.

In this paper, using the path-breaking work of Shilnikov (1965), we find that there may be further reasons to distrust the ability of Taylor rules to be conducive to stability.¹⁰ We show that this policy may induce a class of policy difficulties, emerging from the onset of a chaotic attractor. If the economy becomes enmeshed in a chaotic attractor, the policy maker faces unwanted challenges. Within a chaotic attractor, there is sensitivity to initial conditions, even to infinitesimal changes in initial conditions. Long term predictions become nearly impossible, since an initial condition is known only to a finite degree of precision. It becomes impossible to predict dynamics far into the future. Small changes in initial conditions have major effects on future temporal evolution.

Moreover, given the initial value of the predetermined variable, there would exist a continuum of initial values of the jump variables giving rise to admissible equilibria. Policy options required for recovering uniqueness suggested by the local analysis are exactly those which would cause global indeterminacy of the equilibrium.

Additionally, the qualitative “dimensions” of the chaotic attractor are of great interest in the present context.¹¹ The relative frequency with which an orbit visits different regions of the attractor is heterogeneous. Then, throughout the attractor, the economy lingers on regions with higher “densities.” This is exactly what happens in the numerical simulations developed in this paper. If the initial conditions of the jump variables are chosen far enough from the target steady state, then the emerging aperiodic dynamics continue to evolve over a long period of time around lower-than-targeted inflation and nominal interest rates. This can be interpreted as a liquidity trap phenomenon that, in our case, will depend on the presence of a chaotic attractor and not on the influence of an unintended steady state.¹²

The mathematical underpinnings behind these results exploit the presence of a family of homoclinic orbits, double asymptotic to a saddle-focus, in a three-dimensional ambience. The striking complexity of the dynamics near these homoclinic orbits has been discovered and investigated by Shilnikov (1965), who has shown that, if the associated saddle quantity is positive, infinitely many saddle limit cycles coexist at the bifurcation point. Each of these saddle limit cycles has both stable and unstable manifolds, which determine high sensitivity to initial conditions and irregular transitional dynamics. To the best of our knowledge, the Shilnikov homoclinic bifurcation theorem, largely used in physics, biology, electronic circuits, chemistry and mechanical engineering, has recently found application in economics only in a growth theory paper (Bella, Mattana and Venturi, 2017).

The fourth section of our paper discusses an innovative solution to these unfamiliar problems, if the

¹⁰ Consider, for example, the case in which the policy maker runs an active fiscal-monetary regime. Assume further that a change in the conduct of fiscal policy induces uniqueness of the equilibrium around the intended steady state. Then, the policy maker may be pressured to renounce discretion in fiscal policy by committing to a marginal tax rate above the real interest rate. As we show, a consequence could be Shilnikov chaos.

¹¹Cf. Farmer *et al.* (1983) for a classical discussion on the relevant dimensions of a chaotic attractor.

¹²In contrast, as discussed below in Section 2, Benhabib *et al.* (2001 a,b) found that when the zero bound on nominal interest rate is explicitly taken into account, aggressive interest rate policies may lead the economy to an unintended equilibrium at a liquidity trap or to a limit cycle characterized by Hopf bifurcation. The low inflation rate and low interest rate phenomenon arising in our research, as a consequence of density heterogeneity in the Shilnikov chaotic attractor, is disconnected from the liquidity trap that can emerge because of the influence of an unintended steady state, as in Benhabib *et al.* (2001a,b). In fact, the two types of liquidity trap may even co-exist for a while, depending on the initial conditions of the economy.

Central Bank chooses to retain the Taylor rule and its consequent Shilnikov chaos. Specifically, we show that the chaotic dynamics can be controlled, in the sense of Ott, Grebogi and Yorke (1986), henceforth OGY. Under specific conditions, the announcement of a higher nominal interest rate at the steady state anchors expectations to the long-run target. More generally, the long run nominal interest rate can be treated as an intermediate target of policy, with the instrument being one of the new policy instruments available, such as forward guidance or quantitative easing. Undesired irregular and cyclical behavior can be superseded, and the intended fixed point can be targeted and attained in a relatively short time.

We now present the plan of the paper. The second section presents the model and the implied three-dimensional system of first-order differential equations. We also obtain stability results for the intended steady state, when monetary policy is active. In the third section, we show that the three-dimensional dynamics, characterizing the solution of the model, can satisfy the requirements of the Shilnikov (1965) theorem under plausible calibration settings of the NK model. An example of chaotic dynamics is also discussed, along with its sensitivity to perturbations of the bifurcation parameter and the initial conditions. In section 4 we consider policy approaches to solving the problems produced by the dynamics of the economy within the Shilnikov attractor set. We consider approaches to eliminating the chaos by replacing the Taylor rule by an alternative policy design without interest rate feedback. We also consider approaches that retain the Taylor rule and the associated Shilnikov chaos, while controlling the chaos through the OGY algorithm using a second policy instrument. The conclusion reassesses the main findings of the paper.

2. The model

Consider the optimization problem faced by household-firm i in the sticky-price, money-in-the utility-function, NK model in continuous time (cf., *inter al.* Benhabib *et al.*, 2001a,b; and more recently Tsuzuki, 2016).¹³ We shall call this problem Decision P.

Decision P:

$$\text{Max}_{c_i, m_i, l_i} \int_0^{\infty} \left[u(c_i, m_i) - f(l_i) - \frac{\eta}{2} (\pi_i - \pi^*)^2 \right] e^{-\rho t} dt$$

subject to

$$\dot{a}_i = (R - \pi_i)a_i - Rm(c_i, R) + \frac{p_i}{p} y(l_i) - c_i - \tau$$

$$\dot{p}_i = \pi_i p_i$$

¹³ The money in the utility function approach implicitly uses the derived utility function shown to exist by Arrow and Hahn (1971), if money has positive value in equilibrium. A long literature has repeatedly confirmed this existence from models having various explicit motives for holding money, such as transactions or liquidity constraints (e.g., Feenstra (1986), Poterba and Rotemberg (1987), and Wang and Yip (1992)). Recently, in a dynamical framework, Benhabib, Schmitt-Grohè, and Uribe (2001a,b;2002) have shown equivalence to a money in the production function model.

The mapping from explicit motives for holding money to the derived utility function does not have a unique inverse. Hence, money in the utility function models cannot reveal the explicit motive for holding money. But the ability to infer the explicit motive is not relevant to our research. Hence, for our purposes, we can assume that money has positive value in equilibrium, without conditioning upon an explicit motive.

$$\begin{aligned} a_i(0) &= a_{i0} \\ p_i(0) &= p_{i0} \end{aligned}$$

The objective of the household-firm optimizer is to maximize the discounted sum of a *net* utility stream, where $u(c_i, m_i)$ measures utility derived by household-firm i from consumption of the composite good, c_i , and from real money balances, m_i , under the time discount rate, ρ . It is assumed that $u(\cdot, \cdot)$ is twice continuously differentiable in all its arguments and that

$$u_c(c_i, m_i) > 0; \quad u_{cc}(c_i, m_i) < 0; \quad u_m(c_i, m_i) > 0; \quad u_{mm}(c_i, m_i) < 0, \quad (1)$$

where the function subscripts denote partial derivatives.

The function $f(l_i)$ measures the disutility of labor, where $f(l_i)$ is twice continuously differentiable, with $f_l > 0$ and $f_{ll} < 0$.

The term $\frac{\eta}{2}(\pi_i - \pi^*)^2$ is standard to account for deviations of the price change, $\pi_i = \frac{\dot{p}_i}{p_i}$, with regard to the intended rate π^* , where p_i is the price charged by individual i on the good it produces, and where the parameter η measures the degree to which household-firms dislike to deviate in their price-setting behavior from the intended rate of inflation, π^* .

In the household-firm budget constraint, a_i denotes real financial wealth, consisting of interest-bearing government bonds, where R is the nominal interest rate and $y(l_i)$ is an endowment of perishable goods, produced according to a production function using labor, l_i , as the only input. Real lump-sum taxes are denoted by τ . Therefore, the instantaneous budget constraint says that the change in the firm-household real wealth equals real interest earnings on wealth, plus disposable income net of the opportunity cost of holding money minus consumption expenditure.

Before applying the Maximum Principle, it is important to recall that in the NK model, sales of good i are demand determined,

$$y(l_i) = \left(\frac{p_i}{p}\right)^{-\phi} y^d, \quad (2)$$

where $\phi > 1$ is the elasticity of substitution across varieties, and p is the aggregate price level.

Taking into account (2), the discounted Hamiltonian can be set as

$$\begin{aligned} \mathbf{H} &= u(c_i, m_i) - f(l(p_i)) - \frac{\eta}{2}(\pi_i - \pi^*)^2 + \\ &+ \mu_1 \left\{ [R - \pi_i]a_i - Rm_i + \frac{p_i}{p} \left(\frac{p_i}{p}\right)^{-\phi} y^d - c_i - \tau \right\} + \mu_2 \pi_i p_i, \end{aligned}$$

where μ_1 and μ_2 are the costate variables; c_i and m_i are the control variables; and p_i and a_i are the state variables.

The necessary first order conditions are

$$\frac{\partial H}{\partial c_i} = u_c(c_i, m_i) - \mu_1 = 0, \quad (3.a)$$

$$\frac{\partial H}{\partial m_i} = u_m(c_i, m_i) - \mu_1 R = 0, \quad (3.b)$$

$$\frac{\partial H}{\partial \pi_i} = -\mu_1 a_i + \mu_2 p_i - \eta(\pi_i - \pi^*) = 0, \quad (3.c)$$

$$\dot{\mu}_1 = \rho \mu_1 - \frac{\partial H}{\partial a_i} = \rho \mu_1 - (R - \pi_i) \mu_1, \quad (3.d)$$

$$\dot{\mu}_2 = \rho \mu_2 - \frac{\partial H}{\partial p_i} = \rho \mu_2 + f'(l(p_i))l'(p_i) - (1 - \phi) \frac{y_i l(p_i)}{p} \mu_1 - \mu_2 \pi_i. \quad (3.e)$$

Second order conditions also require

$$u_{cc}(c_i, m_i) < 0, \quad (4.a)$$

$$u_{cc}(c_i, m_i)u_{mm}(c_i, m_i) - u_{cm}(c_i, m_i)^2 > 0. \quad (4.b)$$

Consider now a symmetric equilibrium in which all household-firm units' behaviors are based on the same equations. Then, recalling that in equilibrium

$$c = y(l),$$

the equations from (3.a) to (3.e) allow us to derive the following three-dimensional system of differential equations, which we shall call System *M*.

System *M*:

$$\dot{\mu}_1 = (\rho - R + \pi) \mu_1$$

$$\eta \dot{\pi} = \rho(\pi - \pi^*) \eta - c(\mu_1, \pi) [(1 - \phi) \mu_1 + \phi c(\mu_1, \pi)^\psi]$$

$$\dot{a} = (R - \pi) a - Rm(c(\mu_1, \pi), R) - \tau,$$

where the subscripts are dropped to simplify notation (cf. Tsuzuki, 2016, and Benhabib *et al.*, 2001a,b for details on the derivation). The first equation denotes the time evolution of the Lagrange multiplier associated with the continuous time budget constraint (or shadow price of the real value of aggregate per capita government liabilities, real balances and bonds) at instant of time t .¹⁴ The second equation is the well-known New Keynesian Phillips Curve. The third equation is the budget constraint at time t .

¹⁴ Notice that in the Tsuzuki (2016) formulation of the model, there is also a term representing real government spending, which however is held constant. Since the term has no qualitative relevance for the results in this paper, we neglect it, in line with the Benhabib *et al.* (2001a,b) formulation.

Solutions of system M are admissible equilibrium paths, if the Transversality Condition (TVC)

$$0 = \lim_{t \rightarrow \infty} e^{-\int_0^t [R(s) - \pi(s)] ds} a(t) \quad (5)$$

is satisfied.¹⁵

We now turn our attention to the behavior of the public authorities. Following Benhabib *et al.* (2001a,b), we assume that the monetary authority adopts an interest rate policy described by the feedback rule,

$$R = R(\pi). \quad (6)$$

The function $R(\pi)$ is continuous, strictly convex, and satisfies the following properties.

Assumption 1. (Zero lower bound on nominal rates and Taylor principle). *Monetary authorities set the nominal interest rate as an increasing function of the inflation rate, implying that*

$$R = R(\pi) > 0; \quad R'(\pi) > 0; \quad R''(\pi) < 0. \quad (7)$$

It is further assumed that there exists an inflation rate, π^* , at which the following steady-state Fisher equation is satisfied:

$$R(\pi^*) = \bar{R}. \quad (8)$$

Consider, moreover, the following definition (cf. Benhabib *et al.*, 2001a,b).

Definition 1. *Let $R'(\pi) > 1$. Then the policy maker reacts more than proportionally to an increase in the inflation rate (active monetary policy). If, conversely, $R'(\pi) < 1$, the policy maker reacts less than proportionally to an increase in the inflation rate (monetary policy is passive).*

Let us now turn our attention to fiscal policy. We assume that taxes are tuned according to fluctuations in total real government liabilities, a , so that

$$\tau = \tau(a). \quad (9)$$

Similarly, for monetary policy, it is further assumed that there exists a tax rate corresponding to the steady-state state level of real government liabilities

$$\tau(a^*) = \bar{\tau}. \quad (10)$$

As in Leeper (1991), Woodford (2003), and Kumhof *et al.* (2010), we provide a definition of the fiscal

¹⁵ Notice that, in the present context, the TVC consists of a borrowing limit, preventing households from engaging in Ponzi games.

policy stance. Let us consider the responses of a to its own variations. We have

$$\frac{\partial \dot{a}}{\partial a} = R(\pi) - \pi - \tau'(a). \quad (11)$$

The dynamic path of total government liabilities is locally stable or unstable, according to the magnitude of the marginal tax rate, $\tau'(a)$. Therefore, we have the following useful definition.

Definition 2. *Let $\tau'(a) > R(\pi) - \pi$. Then, since the dynamic path of total government liabilities is stable, fiscal policy is passive. Let $\tau'(a) < R(\pi) - \pi$. Then the dynamic path of total government liabilities is unstable, and the fiscal policy is of active type.*

Notice that adopting a passive fiscal policy is tantamount to committing to fiscal solvency under all circumstances.

2.1. Steady states and local stability properties

The long-run properties of system M are well understood. Benhabib *et al.* (2001a,b) show that if Assumption 1 holds, then, in general, two steady states exist: one where inflation is at the intended rate $\pi = \pi^*$ and one where $\pi = \bar{\pi} \neq \pi^*$.¹⁶ The unintended steady-state is labelled as a liquidity trap, in which the interest rate is zero or near-zero, and inflation is below the target level and possibly negative. Moreover, at the steady-state where inflation is at the intended rate, $\pi = \pi^*$, it follows that μ_1^* exists and is unique.

The local stability properties around the intended steady-state are also well described in the literature. A complete picture is provided in Tsuzuki (2016), where the following are clear.

1. When monetary policy is passive, an active fiscal policy induces uniqueness of the equilibrium. Conversely, a passive fiscal policy commitment to preserve fiscal solvency under all circumstances leads to an indeterminate equilibrium.
2. When monetary policy is active, the stability properties are more mixed. Using the steady state degree of complementarity/substitutability between money and consumption in the utility function, u_{cm}^* , to characterize the results, we have the following.

2a. When money and consumption are Edgeworth complements in the utility function ($u_{cm}^* > 0$), the combination of an active monetary policy regime with a passive fiscal rule still induces uniqueness of the equilibrium. Conversely, no equilibria exist in the neighborhood of the steady state in the case of an active fiscal policy.

2b. When money and consumption are Edgeworth substitutes in the utility function ($u_{cm}^* < 0$), there exists a critical threshold,

$$u_{cm}^* = \hat{u}_{cm}^*, \quad (12)$$

¹⁶ Notice that, as discussed by Benhabib *et al.* (2001a,b), if monetary policy is active according to Definition 1, then $\bar{\pi} < \pi^*$, and the low-inflation equilibrium can be interpreted as a liquidity trap.

such that if $|u_{cm}^*| < |\hat{u}_{cm}^*|$, then the same consequence as in (2a) occurs. Conversely, when $|u_{cm}^*| > |\hat{u}_{cm}^*|$, full stability of the intended steady state is established, when fiscal rule is passive, while indeterminacy of the equilibrium prevails, when fiscal policy is active.

For the sake of a clear discussion of the main point of this paper, we shall assume the following.

Assumption 2. *Money and consumption are Edgeworth substitutes in the utility function, i.e. $u_{cm}^* < 0$.*¹⁷

Let $P^* \equiv (\mu_1^*, \pi^*, a^*)$ denote the values of (μ_1, π, a) such that $\dot{\mu}_1 = \dot{\pi} = \dot{a} = 0$, with

$$\mu_1^* = \frac{\phi}{\phi - 1} c(\mu_1^*, \pi^*)^\psi$$

$$\pi^* = \bar{R} - \rho,$$

$$a^* = \frac{\bar{R}m(c(\mu_1^*, \pi^*), \bar{R}) + \bar{\tau}}{\rho},$$

defined as the intended steady state. Then, we prove the following result.

Proposition 1. (Local stability properties of the intended steady state under Assumption 2). *Recall Assumption 2. Assume monetary policy is active. Then two stability cases can occur according to the magnitude of $|u_{cm}^*|$. Consider, first, the case, $|u_{cm}^*| < |\hat{u}_{cm}^*|$. If fiscal policy is also active, P^* is a repeller and there are no equilibrium paths besides the steady-state itself. If fiscal policy is passive, P^* is a saddle of index 2, and the equilibrium is locally unique. Consider now the case, $|u_{cm}^*| > |\hat{u}_{cm}^*|$. If fiscal policy is passive, P^* is an attractor, whereas when fiscal policy is active, there is a continuum of equilibria that converge to the steady-state (local indeterminacy).*

Proof. These results are obtained by applying the Routh-Hurwitz stability criterion to system M , evaluated at the steady state. Cf. Appendix 1. ■

¹⁷ We are looking for parameter combinations such that system M possesses a hyperbolic, saddle-focus equilibrium point. Therefore, both cases (2.a) and (2.b) are possible candidates. However, as will become clearer later on in the paper, it is more convenient to spotlight the (2.b) case, where the saddle-focus may bifurcate into a fully stable equilibrium point. The economic implications of a negative u_{cm} are well represented in Walsh (2010) for the general case of the utility function with non-zero interdependences between leisure, money, and consumption. Specifically, if $u_{cm} < 0$, a monetary injection that raises expected inflation will increase consumption, labor supply, and output, a situation described as an “asset substitution model” by Wang and Yip (1992).

Since Edgeworth substitutability is a cardinal property, it is not econometrically testable. But closely related Morishima substitutability is ordinal and has been tested by Serletis and Xu (2019). They found (see their figure 11, p. 21) that consumer goods have consistently been net Morishima substitutes for monetary services throughout their sample period, beginning in 1967, but gross complements because of positive income effects. Since income effects are not relevant to Edgeworth substitutability, the finding of net Morishima substitutability is more relevant to our assumption. Consumer goods might be both net and gross substitutes for monetary services, if monetary services are augmented to include credit card services, as available with the Divisia monetary aggregates supplied by the Center for Financial Stability. Increased consumption is associated with increased use of credit cards.

3. Shilnikov chaos

Let us now focus on the case, $|u_{cm}^*| < |\hat{u}_{cm}^*|$. Consider a scenario where the policy-maker runs an active fiscal-monetary regime. Then, by Proposition 1, the policy maker may be pressured to increase the marginal tax rate above the real interest rate. In this Section, we show that following this policy prescription may induce another class of difficulties.

3.1. An explicit variant of the model

Before proceeding with our analysis, we need to provide specific forms for the implicit functions presented in system M . Following the standard literature, we first assume that the utility function has constant relative risk aversion in a composite good, which in turn is produced with consumption goods and real balances via a CES aggregator as follows:

$$u(c, m) = \frac{[\kappa c^{1-\beta} + (1-\kappa)m^{1-\beta}]^{\frac{1-\Phi}{1-\beta}}}{1-\Phi}, \quad (13)$$

where $0 < \kappa < 1$ is a share parameter, β measures the intra-temporal elasticity of substitution between the two arguments, c and m , and $\Phi > 0$ is the inverse of the intertemporal elasticity of substitution. Since we have, for now, assumed that consumption and real money balances are Edgeworth substitutes, the following parametric restriction is implied.

Remark 1. $Sign(u_{cm}^*) = Sign(\beta - \Phi)$. Therefore, Assumption 2 requires $\beta < \Phi$.¹⁸

Moreover, it is standard to assume that the disutility of labor is captured by the following functional form

$$f(l) = \frac{l^{1+\psi}}{1+\psi}, \quad (14)$$

where $\psi > 0$ measures the preference weight of leisure in utility.

Furthermore, following Carlstrom and Fuerst (2003) and more recently Tsuzuki (2016), we also assume that production is linear in labor,

$$y(l) = Al, \quad (15)$$

where A denotes the productivity level in the composite goods production. Without loss of generality, we will also set $A = 1$.¹⁹

¹⁸ Cf. Walsh (2010), p. 72, for an extensive discussion of the economic interpretation of this restriction.

¹⁹ Carlstrom and Fuerst (2003) make the same assumption.

Additionally, we use the specification of the Taylor principle in Benhabib *et al.* (2001a,b), and assume that monetary authorities observe the inflation rate and conduct market operations to ensure that

$$R(\pi) = \bar{R}e^{(C/\bar{R})(\pi-\pi^*)}, \quad (16)$$

where C is a positive constant. Notice that, from (6), our chosen functional form implies

$$R(\pi^*) = \bar{R}; \quad R'(\pi^*) = C. \quad (17)$$

Finally, in order to avoid violating the Transversality Condition, we assume that the economy satisfies a Ricardian monetary-fiscal regime. More specifically, equation (16) is complemented by the fiscal rule

$$\tau(a) = \alpha a - Rm, \quad (18)$$

where the marginal tax rate $\alpha \equiv \tau'(a)$ is a positive constant.

3.2. Conditions for the existence of Shilnikov chaos

In this section, we provide the mathematical underpinnings that guarantee the existence of a chaotic regime in system M . Consider the following Theorem (Chen and Zhou, 2011), which is a generalized version of the original result of Shilnikov (1965).

Theorem 1. *Consider the dynamic system*

$$\frac{dY}{dt} = f(Y, \alpha), \quad Y \in \mathbb{R}^3, \quad \alpha \in \mathbb{R}^1,$$

with f sufficiently smooth. Assume f has a hyperbolic saddle-focus equilibrium point, $Y_0 = 0$, at $\alpha = 0$, implying that eigenvalues of the Jacobian, $J = Df$, are of the form γ and $\chi \pm \xi i$, where γ , χ , and ξ are real constants with $\gamma\chi < 0$. Assume that the following conditions also hold:

- (H.1) *The saddle quantity, $\sigma \equiv |\gamma| - |\chi| > 0$;*
- (H.2) *There exists a homoclinic orbit, Γ_0 , based at Y_0 .*

Then the following results hold:

- (1) *The Shilnikov map, defined in the neighborhood of the homoclinic orbit of the system, possesses an infinite number of Smale horseshoes in its discrete dynamics;*
- (2) *For any sufficiently small C^1 -perturbation, g , of f , the perturbed system has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map, defined in the neighborhood of the homoclinic orbit;*
- (3) *Both the original and the perturbed system exhibit horseshoes chaos.*

The application of Theorem 1 to system M requires that several conditions be fulfilled, gradually restricting the relevant parameter space. Specifically, we need the parameters to be such that: (i) the system possesses a hyperbolic saddle-focus equilibrium point; (ii) in the case of the saddle-focus equilibrium, the saddle quantity is positive; and (iii) in the case of the saddle-focus equilibrium, with positive saddle quantity, there exists a homoclinic orbit connecting the saddle-focus to itself. System M is highly non-linear and heavily parametrized.

Our attempts to obtain a general result on the critical parametric bifurcation surfaces have been frustrated by frequent numerical anomalies. In order to show that there are regions in the parameter space such that system M may satisfy the conditions of Theorem 1, we therefore propose a numerical strategy based on the parametrization of the US economy for the period 1960(Q1) to 1998(Q3), proposed by Benhabib *et al.* (2001a,b) and extensively used in the succeeding literature (cf. Tsuzuki, 2016, for a recent application).

Example 1. Denote the set of the deep parameters as $\mathbf{D} \equiv (\beta, \eta, \kappa, \phi, \psi, \rho, \Phi)$, and assume

$$\bar{\mathbf{D}} \equiv (1.975, 350, 0.90899, 21, 1, 0.018, 2) \in \mathbf{D}.$$

Set furthermore the pair $(\bar{R}, \pi^*) = (0.06, 0.042)$ to match the (average) three-month Treasury Bill rate and (average) inflation rate observed over the period for the US economy. Therefore, since τ cancels out in the calculations, the characteristic equation (A.2 in Appendix 1) is a function of the remaining policy parameters, C and τ' . Solving the characteristic equation gives

$$\begin{aligned} \lambda_1 &= 0.018 - \tau', \\ \lambda_{2,3} &= 0.009 - 0.00058C \pm 0.00058\sqrt{(C - 1.00046)(C - 4.6543 \times 10^5)}. \end{aligned}$$

Therefore, since $|u_{cm}^*| \cong 0.0008 < |\hat{u}_{cm}^*| \cong 15.7812$ according to Proposition 1, an active monetary-fiscal regime implies three eigenvalues with positive real parts for any reasonable value of the coefficient C . A fiscal policy switch to a passive rule implies one negative eigenvalue and two eigenvalues with positive real parts. In this example, the saddle quantity equals

$$\sigma \equiv \tau' - 0.027 + 0.00058C. \tag{19}$$

Therefore, if we set $\tau' > 0.027 - 0.00058C$, the saddle quantity is positive.

We are now ready to propose the following result.

Lemma 1. (Fulfillment of pre-condition H.1 in Theorem 1). *There are regions of the parameter space where the intended steady-state, P^* , is a saddle-focus equilibrium with $\sigma > 0$.*

Proof. Set $C > 1.00046$ and $\tau' > 0.027 - 0.00058C$ as in Example 1. Then, the eigenvalues associated with system M are of the form required for P^* to be a saddle-focus equilibrium with $\sigma > 0$.

■

In order to verify the robustness of the results in Lemma 1 to changes in the parameters, we conducted some further numerical simulations. First, we obtained a more general form of the eigenvalues by relaxing the parameters one-by-one from the set $\mathbf{D} \equiv (\beta, \eta, \kappa, \phi, \psi, \rho, \Phi)$. Results, not reported but available upon request, indicate that there is always a small range of the parameter C above unity, for which eigenvalues are all real and there exist values of the marginal tax rate, such that $\sigma > 0$.

Furthermore, we kept $\mathbf{D} = \bar{\mathbf{D}}$, took $\pi^* = 0.042$, and studied the surface

$$\Omega \equiv \mathbf{B}(\mathbf{J}) + \text{Tr}(\mathbf{J})^2$$

in the remaining (C, \bar{R}, τ') parameter space. As shown in Bella, Mattana and Venturi (2017), the vanishing of Ω corresponds to the critical parametric surface at which a generic steady state is a saddle-focus equilibrium with null saddle quantity.

Figure 1. *Combinations of the (C, \bar{R}, τ') parameters at which P^* is a saddle-focus with $\sigma = 0$.*

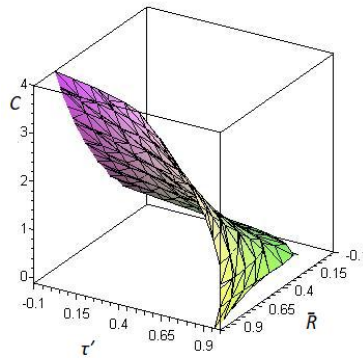


Figure 1 depicts the parametric surface in the (C, \bar{R}, τ') space such that P^* is a saddle-focus equilibrium at the bifurcation point $\sigma = 0$. Above the surface, the saddle quantity is positive. Below the surface, the saddle quantity is negative. Interestingly, the figure shows that a positive saddle quantity can be determined exactly, when the pair (\bar{R}, τ') is plausibly low and $C > 1$.

We end this section by noticing some further details regarding the form of the eigenvalues in Example 1. It is clear that, for $C > 1.00046$, and irrespective of the stance of fiscal policy, one eigenvalue is real, and the remaining two eigenvalues are complex conjugate. This means that locally, when monetary policy is active, convergence towards P^* occurs typically through (damped) oscillating paths.²⁰ It is also useful to observe the following.

Remark 2. *In our simulations, the structure of the eigenvalues derived in Example 1 survives wide variations of the parameters. More specifically, when $C > 1$ (active monetary policy), there is always a*

²⁰ It would be interesting here to confront the dynamics featured by these equilibria with the “volatile sequence of interest rates and inflation rates followed by sudden arrival at the low nominal interest rate steady state,” pointed out by Bullard (2010, p. 344), regarding complicated or chaotic expectational dynamics. On that issue, see also Piazza (2016).

small right neighborhood of $C = 1$ such that eigenvalues are all real. Things are different when $C < 1$ (passive monetary policy). In this case, eigenvalues are always real, and the convergence towards P^* generally takes place along the monotonic perfect-foresight path.

Once it has been established that there are regions in the parameter space such that P^* is a saddle-focus equilibrium with $\sigma > 0$, we need to show that system M admits homoclinic solutions (pre-condition $H.2$ in Theorem 1). Bella, Mattana and Venturi (2017) describe in detail the necessary steps required to establish whether a given dynamical system supports the existence of a family of homoclinic orbits doubly asymptotic to a saddle-focus in \mathbb{R}^3 . The application of the method is very lengthy. Because of the space constraint, we do not report those computations, which remain available upon request. For details, please refer to Bella, Mattana and Venturi (2017).

A preliminary step requires translation of the system, M , to the origin and putting the system into normal form by using the associated eigenbasis. We thereby obtain the following (truncated) normal form of system, M ,

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} \chi & -\xi & 0 \\ \xi & \chi & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} F_{1a}w_1w_2 + F_{1b}w_1w_3 + F_{1c}w_2w_3 + F_{1d}w_1^2 + F_{1e}w_2^2 + F_{1f}w_3^2 \\ F_{2a}w_1w_2 + F_{2b}w_1w_3 + F_{2c}w_2w_3 + F_{2d}w_1^2 + F_{2e}w_2^2 + F_{2f}w_3^2 \\ F_{3a}w_1w_2 + F_{3b}w_1w_3 + F_{3c}w_2w_3 + F_{3d}w_1^2 + F_{3e}w_2^2 + F_{3f}w_3^2 \end{pmatrix}, \quad (20)$$

where $(w_1, w_2, w_3)^T$ is the vector of transformed coordinates, and where the $F_{i,j}$ coefficients, with $i = 1, 2, 3$ and $j = a, b, \dots, f$, are combinations of the original parameters of the model, also depending on the values of three free constants, φ_i , $i = 1, 2, 3$ arising in the computation of the eigenbasis. Following Freire *et al.* (2002), system (20) can be put into the hypernormal (truncated) form

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ dw_1^2 + kw_1^3 \end{pmatrix}, \quad (21)$$

where $\varepsilon_1 = -\mathbf{Det}(\mathbf{J})$, $\varepsilon_2 = \mathbf{B}(\mathbf{J})$, $\varepsilon_3 = -\mathbf{Tr}(\mathbf{J})$, and where d and k are combinations of various $F_{i,j}$ coefficients.

Once the hypernormal form has been obtained, the method of undetermined coefficients (Shang and Han, 2005) is applied to obtain a polynomial approximation of the analytical expressions of both the two-dimensional unstable manifolds associated with λ_2 and λ_3 , and of the one-dimensional stable manifold associated with λ_1 . The procedure leads to the following split function²¹

$$\Sigma = \Xi + \frac{F_{3f}\Xi^2}{\gamma} + (2\chi - \gamma) \frac{F_{3a}\Psi\Omega + F_{3d}\Psi^2 + F_{3e}\Omega^2}{(2\chi - \gamma)^2 + 4\xi^2} = 0, \quad (22)$$

²¹ Cf. Kuznetsov (1998, p. 198) for the geometrical interpretation of the split function in the context of homoclinic bifurcations.

where $(\mathcal{E}, \Psi, \Omega) \in (0,1)^3$ are free constants, while $\gamma = \lambda_1$, $\chi = \text{Re}(\lambda_{2,3})$, and $\xi = \text{Im}(\lambda_{2,3})$. Then, with given parameters, conditions for the existence of the homoclinic loop, doubly asymptotic to the saddle-focus equilibrium point, rely on the existence of a triplet $(\mathcal{E}, \Psi, \Omega) \in (0,1)^3$ satisfying $\Sigma = 0$ (admissible solution).²²

To verify whether there are admissible solutions to (22) in the feasible parameter space, we specify further the calibration of the economy used in this section.

Example 2. Let $\mathbf{D} = \bar{\mathbf{D}}$ and $(\bar{R}, \pi^*) = (0.06, 0.042)$, as in Example 1. Set $\tau = 1.5$. By (19), the critical value of the marginal tax rate at which the saddle quantity is positive is

$$\tilde{\tau}' = 0.027 - 0.00058C = 0.02613.$$

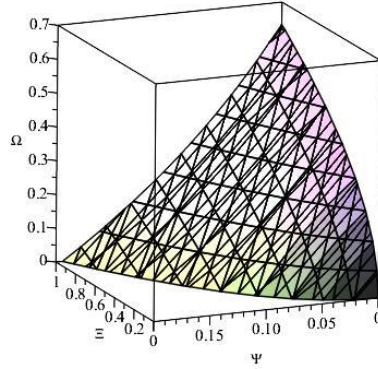
If $\tau' > \tilde{\tau}'$, then P^* is a saddle-focus with positive saddle quantity. Let us now use the marginal tax rate, τ' , as the bifurcation parameter. More precisely, we iteratively increase τ' above 0.02613 with a grid of 0.01 until a solution for $\Sigma = 0$ with $(\mathcal{E}, \Psi, \Omega) \in (0,1)^3$ emerges. The procedure reveals that there exists an interval $I_{\tau'} \cong (0.02613, 0.23543)$ such that, for all $\tau' \in I_{\tau'}$, a family of homoclinic loops doubly asymptotic to the saddle-focus equilibrium point exists.²³

Figure 2 depicts the combinations of the $(\mathcal{E}, \Psi, \Omega)$ constants solving the split function (22) for the case of $\tau' = 0.15$.

Figure 2. Coordinates in the $(\mathcal{E}, \Psi, \Omega)$ space giving rise to the homoclinic loop for $\tau' = 0.15$.

²² The reason why the three constants $(\mathcal{E}, \Psi, \Omega)$ are bound to belong to the cube $(0,1)^3$ is strictly related to the geometry of the stable and unstable manifolds which intersect near the origin (in the transformed eigenspace) and forms the homoclinic loop. The issue is well identified in Kuznetsov (1998, p. 259).

²³ In order to identify the monetary policy and fiscal policy regimes that prevailed in the US, Bhattarai *et al.* (2012) considered 90 percent prior probability interval for the parameter C to be (1.189, 1.811) under active monetary policy regimes and for marginal tax rates to be (0.003, 0.107) under passive fiscal policy regimes in their calibrations. The intervals cover the range of values found in the literature (e.g., Davig and Leeper (2011), Xu and Serletis (2016), Ascari *et al.* (2017) etc.). Using more recent US data with a superior policy rule that incorporates time varying disturbance variances in interest rate rules, Xu and Serletis (2015) found parameter $C = 1.655$ and marginal tax rate $\tau' = 0.017$ under the active-passive monetary-fiscal regime. However, note that the extent to which the marginal tax rate can be revised upwards depends on where the economy is located on its Laffer curve and the political resistance to higher taxes on the economy.



Remark 3. For this calibration of the economy, since $\frac{\partial \Sigma}{\partial \tau'} > 0$ for all values of $(\Xi, \Psi, \Omega) \in (0,1)^3$, there exists a unique critical value of τ' solving the split function (22).

The following statement is therefore implied.

Lemma 2. (Fulfillment of pre-condition H.2 in Theorem 1). *There exists regions in the parameter space such that pre-conditions H.1 and H.2 in Theorem 1 are simultaneously satisfied.*

Lemma 2 is an important result. It states that, if the parameters are accurately chosen, then a family of homoclinic connections is established, leading from the intended steady state to itself in backward and forward time.

Without showing the computations, we also point out the following.

Remark 4. *Alternative calibrations of the economy show that the result is qualitatively robust.*

3.3. Existence and properties of the chaotic attractor

We can now go to the main result in this section. Let $\nu = \tau' - \bar{\tau}'$, where $\bar{\tau}' \in I_{\tau'}$ is the critical value of the marginal tax rate, such that an admissible solution of the split function exists for given coordinates, $(\Xi, \Psi, \Omega) \in (0,1)^3$. Let $V \subset \mathbb{R}$ be a small open neighborhood of $\mathbf{0}$. We have the following result.

Proposition 2. (Existence of a Shilnikov chaotic attractor) *Assume that the parametric conditions in Lemmas 1 and 2 are satisfied. Let $\nu \in V$. Then, given a triplet of initial conditions, $(w_1(0), w_2(0), w_3(0))$, sufficiently close to the origin, system (21) admits perfect-foresight chaotic equilibrium solution. By topological equivalence, the result also applies to system M.*

Proof. See Appendix 2. ■

Consider now the following Example.

Example 3. *Set $\mathbf{D} = \bar{\mathbf{D}}$, $(\bar{R}, \pi^*) = (0.06, 0.042)$, and $C = 1.5$, as in Example 2. Then, we know that there exists $I_{\tau'} \cong (0.02613, 0.23543)$ such that for all $\tau' \in I_{\tau'}$ there exists a family of homoclinic loops doubly asymptotic to the saddle-focus equilibrium point. Consider the case of $\tau' = 0.15$ and*

initial conditions $(w_1(0), w_2(0), w_3(0)) = (-0.01, -0.01, -0.01)$.

The attractor generated by this specific example is represented in Figure 3.

Figure 3. *The chaotic attractor in the (w_1, w_2, w_3) space.*

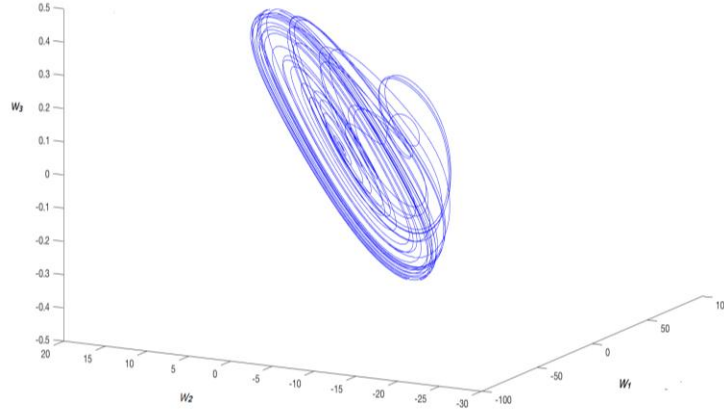


Figure 3 displays the distinct shape of the Shilnikov attractor. The dynamics of the economy along the spiral attractor have periods of relative quiescence, when the phase point approaches the saddle-focus point. Conversely, when the phase point starts to spiral away from the saddle-focus point, there is the onset of irregular episodes of oscillatory activity.²⁴ More details on the characteristics of the time profiles of the variables, when the economy evolves along the chaotic attractor, are provided in the next sections.

3.4. Economic implications

Economic implications of Proposition 2 are very important to the dynamics implied by NK models. The existence of a chaotic attractor implies that small changes in initial conditions can produce large changes in dynamics over time. Two economies, starting contiguously in the space of initial conditions,

²⁴ Is the chaotic attractor in Figure 3 a global absorbing set in which trajectories fall over time for any initial data, or does the considered attractor have only a bounded basin of attraction? A way to answer these questions is to follow Bella, Mattana, and Venturi (2017) and perform a numerical scanning of the initial conditions space in which the attractor is observed. However, since initial conditions are given in the transformed eigenspace $(w_1(0), w_2(0), w_3(0))$, and since it is interesting to understand the boundaries of the basin of attraction in the original (μ_1, π, a) coordinates, we retrace the transformation matrix for the case of parameters as in Example 3. If we consider inflation, we have

$$\pi(0) = \pi^* - \frac{0.00876w_1(0) - 0.18366w_2(0)}{250}$$

where the weights in the formula depend on the structure of the chosen eigenvectors. Applying the iterative procedure, and starting from the vector $(w_1(0), w_2(0), w_3(0)) = (-0.01, -0.01, -0.01)$ in Example 3, we find that the attractor survives any variation of $\pi(0) \in (0.0351, 0.0437)$. This means that any perfect foresight path originating in this interval for inflation, is captured by the chaotic attractor. Notice that, these findings imply that the region of the phase space around the homoclinic orbit, which also belongs to the basin of attraction of the chaotic set, is very narrow. Therefore, as is customary in the literature discussing the characteristics of Shilnikov chaos, it suffices for a small perturbation of the system to make the attractor disappear.

can follow completely different patterns over time. Since an initial condition is known only to a finite degree of precision, it is impossible to predict dynamics deterministically over extended periods of time.

Moreover, within a chaotic attractor, given the initial value of the predetermined variable, there exists a *continuum* of initial values of the jump variables giving rise to admissible equilibria. Therefore, the policy options required to recover the uniqueness suggested by the local analysis are exactly those which may cause global indeterminacy of the equilibrium. In this regard, showing that the equilibrium is globally indeterminate requires the proof that, given an initial condition in terms of the predetermined state variable, $a(0)$, there exist multiple choices of the jump variables, $\mu_1(0)$ and $\pi(0)$, lying outside the small neighborhood relevant to the local analysis. Our analysis is able to give rise to recurrent equilibria, namely solution trajectories of system M that stay in a fixed tubular neighborhood of a given homoclinic orbit for all times.

Let

$$\nu_h = \{v \in V: \text{System } M \text{ exhibits horseshoe chaos}\}.$$

Then, if $v \in \nu_h$, there exists an ε – tubular neighborhood of the homoclinic orbit, such that system M exhibits horseshoe chaos. Let us now denote by $T_v \subset \mathbb{R}^3$ the set of all points of this ε -tubular neighborhood, with $\text{Int } T_v$ and $\text{Bd } T_v$ as the set of all interior and boundary points of T_v , respectively.

Let

$$E_v = \{(\mu_1, \pi, a) \in \mathbb{R}^3: (\mu_1, \pi, a) \in \text{Int}T_v\}$$

be a three-dimensional manifold containing the set of all possible paths starting on $\text{Int } T_v$. Then, by Theorem 1, all paths starting on the (compact) set E_v are recurrent paths, bound to stay forever in E_v .

Consider therefore the following result.

Corollary 1. (Global indeterminacy of the equilibrium). *Assume that the dynamics generated by the flow of system M are of the type discussed in Proposition 2. Let furthermore $v \in \nu_h$. Consider an initial value $a(0) \in E_v$. Then, the NK model exhibits global indeterminacy of the equilibrium.*²⁵ ■

Proof. See Appendix 3. ■

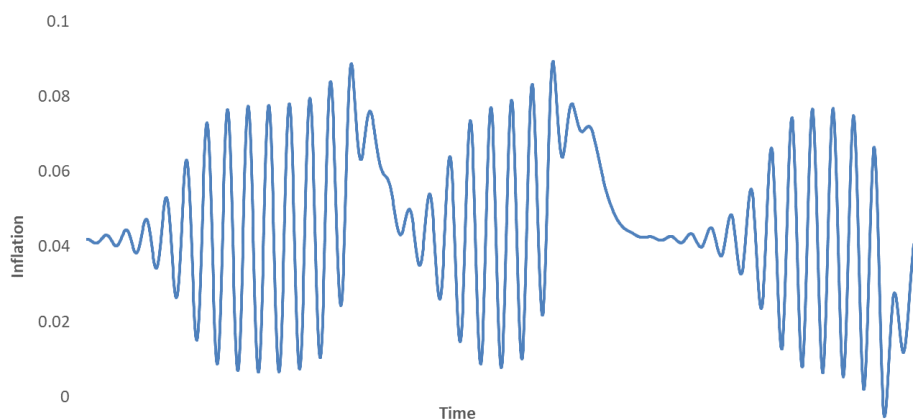
Finally, the qualitative “dimensions” of the chaotic attractor are of particular interest in the present

²⁵ Barnett and Duzhak (2008, 2010, 2019) found Hopf bifurcation boundaries and Period Doubling (Flip) bifurcation boundaries in discrete time NK models. Benhabib *et al.* (2001a,b) and Tsuzuki (2016) located the Hopf bifurcation boundary in the continuous time version of the NK model. These bifurcation boundaries in the NK model parameter space represent different qualitative dynamics within the class of such models. Bifurcation boundaries are in fact commonly found in the parameter spaces of all credible macroeconomic models, such as optimal growth models and overlapping generations models. See, e.g., Grandmont (1985), Geweke, Barnett and Shell (1989), Barnett and Chen (2015); Barnett and Ghosh (2014); and Bella, Mattana and Venturi, (2017).

context.²⁶ Assume that the relative frequency at which an orbit visits different regions of the attractor is largely heterogeneous. Then, across the volume of all possible coordinates contained in the attractor, the economy lingers on particular regions with higher “density.” In the numerical simulations developed in the paper, it is evident that if the initial conditions of the jump variables are chosen far enough from the target steady-state, then the emerging aperiodic dynamics tend to evolve for a long time around lower-than-targeted inflation and nominal interest rates. This can be interpreted as a liquidity trap phenomenon, which now depends on the existence of a chaotic attractor and not on the influence of an unintended steady state.

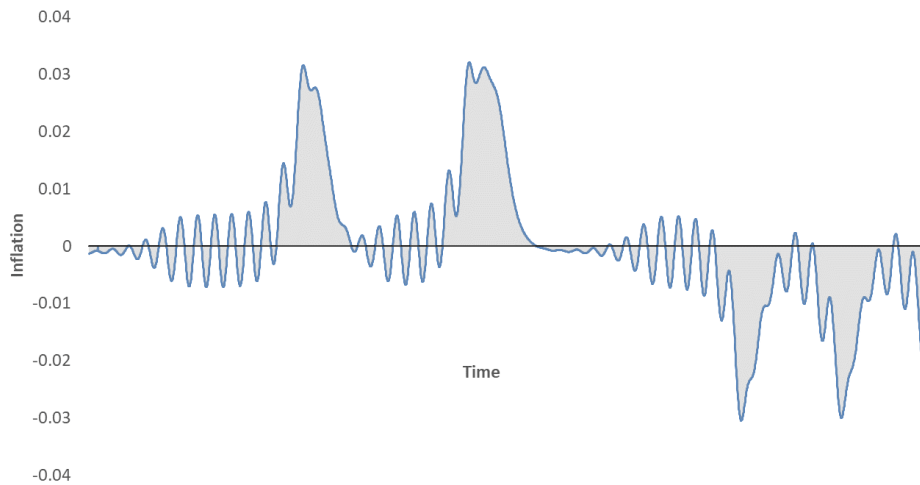
Consider the reconstructed time profile of the inflation rate (Figure 4a). The time span is 10 years. First, observe the distinct shape of Shilnikov chaos. The wave train generated by the spiral attractor has long quiescence periods, when the phase point approaches the saddle-focus, followed by bursts of oscillatory activity. The average inflation rate can be persistently higher/lower than the steady-state value. This implies the possibility of long periods, during which inflation is stubbornly high, or long periods during which inflation is stubbornly low (akin to a deflationary equilibrium), and periods during which inflation is volatile.

Figure 4a. *The time profile of the chaotic inflation rate.*



²⁶Cf. Farmer *et al.* (1983) for a classical discussion on the relevant dimensions of a chaotic attractor.

Figure 4b. *The moving average of the chaotic inflation rate (window = 100 iterations).*



The relevance of periods of persistently low inflation rates is strengthened by a further phenomenon. The time profile for inflation shows a slight negative drift.²⁷ Consider in particular Figure 4b, depicting the moving average of the inflation rate (window = 100 iterations), which we detrend using the steady state value of 0.042. The figure reveals a robust and persistent downward deviation of the moving average from the (de-trended) target inflation rate.

These empirical characteristics of the dynamic pattern imply the following statement.

Corollary 2. (The existence of persistent inflationary/deflationary perfect-foresight equilibrium paths). *Assume that the dynamics of the system evolve along the attractor set. Then, persistently high/low inflation rates with regard to the (unique) steady-state value can emerge.*²⁸

Corollary 2 has important implications for the debate regarding liquidity traps. As discussed in our introduction, this phenomenon has previously been linked mainly to the existence of a low-inflation steady state (cf., in particular, Benhabib *et al.*, 2001a,b) and to its basin of attraction. We offer an alternative explanation, based on the long-run peculiarities of a chaotic attractor and the evolution of the dynamics within that attractor set, such that the economy drifts into the liquidity trap without any policy intent.²⁹

²⁷ We see that, in the very long-run, the dynamics of the inflation rate, in this numerical experiment, settle down to irregular dynamics, not centered on the coordinates of the saddle-focus.

²⁸ Notice that the eigenvalues can be chosen such that inflation can also be persistently negative.

²⁹ Notice that Corollary 2 is based on the empirical observation of the time profile of inflation. However, to provide more formal arguments, we have also computed the generalized Hurst exponent (Hq) for the inflation time series implied by the chaotic attractor emerging from Example 3 (cf. *inter al.* Peters (1991) for the related formula). The exponent, directly related to the fractal dimension, is a measure of “mild” or “wild” randomness of the data series. It quantifies the relative tendency of a time series either to regress strongly to the mean, or to cluster in a direction. When $0.5 < Hq < 1$, we have evidence of a time series with long-term positive autocorrelation, such that a high value in the series will probably be followed by another high value, and values far into the future will also tend to be high. Conversely, when $0 < Hq < 0.5$, we have evidence that a single high value will probably be followed by a low value, and the value after that will tend also to be

The differences in the qualitative dynamics arising because of an unintended steady state or because of the existence of a chaotic attractor are remarkable. The time profile for inflation featured in Benhabib *et al.*, (2001a,b) for the case of $u_{cm} < 0$, so that consumption and real balances are substitutes, presents higher and higher amplitude oscillations around the active steady. Then inflation suddenly arrives to the passive (lower) steady state value, when the saddle connection is established. This kind of predictable/regular behavior of the economy could be traced out by an econometric exercise. In our case, inflation, along the spiral attractor, has long quiescence periods, possibly characterized by a persistent and steep monotonic behavior, followed by bursts of irregular oscillatory activity.

This kind of pattern is largely unpredictable and cannot be inferred by conventional econometric tools, since such behavior violates the regularity conditions for available statistical inference methodologies, such as the usually assumed properties of the likelihood function and polyspectra (see, e.g., Barnett, Gallant, Hinich, Jungeilges, Kaplan, and Jensen (1997) and Geweke (1992)).

As a check on the robustness of our conclusions to our assumption of money in the utility function, we repeat our analysis of appearance of Shilnikov chaos to the alternative specification of money in the production function in Appendix 6.

4. Policy solutions

4.1. Ending the chaos

Potential policy solutions to the problems produced by Shilnikov chaos can be divided into two groups. One group of approaches ends the Shilnikov chaos by removing the Taylor rule and its closed-loop interest rate feedback dynamics, while introducing a fundamentally different monetary policy design. The other approach is to retain the imposed Taylor principle and thereby the Shilnikov chaos, while imposing an algorithm to control the chaos. This latter approach requires introduction of a second policy instrument in addition to the interest rate that appears in the Taylor rule.

To end the chaos, the Central Bank could adopt any of the policy approaches that do not use a Taylor rule. There are many such policy designs in the literature that best could be selected by the Central Bank in accordance with the Central Bank's mechanism design. It is not the purpose of this paper to advocate any one of those alternatives.

Examples could include using an active fiscal policy and a passive monetary policy. That approach produces its own dynamical problems in a NK Model, but not Shilnikov chaos. Another example could include monetary policy without interest rate feedback. An open loop fixed monetary quantity growth rate would be the simplest approach. More sophisticated modern approaches could include those using Divisia monetary quantity aggregates, such as those proposed by Belongia (1996), Serletis (2013), or Belongia and Ireland (2014, 2017, 2018) and as advocated by Peter Ireland in his role on the Shadow Open Market Committee. A long literature exists on alternatives to Taylor rule policies, such as Cochrane (2011).

high. This tendency to switch between high and low values is also expected to last a long time into the future. Our computation (using 4000 iterations) leads to $Hq \approx 0.880417$, which suggests high persistence in the time series.

4.2. Controlling the chaos

If the Central Bank were to decide to retain imposition of the Taylor Principle and thereby the resulting Shilnikov chaos, a second instrument of policy would need to be introduced to deal with the consequent liquidity trap. The need for such a second instrument, in addition to the interest rate in the Taylor rule, is widely accepted and has been applied by most central banks in recent years. A survey of such new tools of monetary policy, such as forward guidance and quantitative easing, has been provided by Bernanke (2020) in his Presidential Address to the American Economic Association. But what is less well established is how to design a rule for use of such an alternative instrument of policy. Under those circumstances, we would propose one of the available algorithms for controlling chaos. In addition, we find that when the economy is on a Shilnikov chaos attractor set associated with imposition of a Taylor rule, the need for a second instrument of policy to control the undesirable properties of chaos should be retained, even if the economy is not in a liquidity trap.

We now consider policy to control chaos. Assume the economy is enmeshed in a chaotic attractor. What should a policy maker do in order to alleviate the implied economic uncertainty, and bring agents' inflationary expectations back in line with those coherent with the intended steady state? In each such approach, one of the new tools of policy would be adopted as a second instrument of policy to target, as an intermediate target, a long run anchor consistent with an available algorithm for controlling chaos.

The methods of controlling chaotic dynamics in the engineering literature provide useful tools in this regard. Under certain conditions, undesired irregular or even cyclical behavior can be switched off.³⁰ An ingenious and well-known method for doing so is proposed by Ott, Grebogi and Yorke (1990), also known as the OGY algorithm. It enables one to force a chaotic trajectory onto a desired target (a periodic orbit or a steady state of the system) by a correction mechanism. This mechanism has the form of a small, time-dependent perturbation of a certain control parameter. Suppose also that a neighborhood of the desired fixed point can be found, such that the system is guaranteed to be driven to the fixed point. If this neighborhood has points in common with a chaotic attractor, it may be used as a controllable target for the fixed point.

To achieve control over a chaotic solution trajectory, the control parameter must be accessible to the Central Bank. This is a relevant point in our NK sticky-price model with Taylor rule interest rate feedback and a Ricardian fiscal policy. If we go back to the form of the eigenvalues in Example 1, it is clear that fiscal policy parameters can only govern the sign of the real eigenvalue.³¹ As a consequence, fiscal policy is ineffective in controlling chaos in the present setting.³²

³⁰ There are examples of chaos control in the literature on optimal monetary models (cf., *inter al.*, Mendes and Mendes, 2006). However, those examples are developed in a discrete-time environment. To the best of our knowledge, this is the first attempt in economic theory with continuous-time dynamics. Experiments of chaos control are widespread in other fields of the economics literature. For example, there are the very recent contributions in tâtonnement processes (cf. *inter al.* Naimzada and Sordi, 2017) and in disequilibrium macroeconomic models (Kaas, 1998).

³¹ Recall that τ cancels out in the computation of the characteristic equation. See also (A.2), (A.3), and (A.4) in Appendix 1.

³² A similar problem is discussed in Benhabib, Schmitt-Grohè, and Uribe (2002). The authors describe the characteristics of fiscal policy schemes capable of eliminating the liquidity trap, while maintaining the assumed monetary policy stance. More

In this regard, we choose to operate through the manipulation of the nominal interest rate, \bar{R} , at which the steady-state Fisher equation in (8) is satisfied. Since it is not under the direct control of the Central Bank, our algorithm treats manipulation of that interest rate as an intermediate target, rather than as an instrument of policy.³³

Before proceeding with the implementation of the OGY algorithm, some preliminary steps need to be taken. First, we need to show that system M is controllable. Then, we will need to discuss the region of the parameter space supporting application of the OGY algorithm. Consider the following initial result.

Lemma 3. *System M satisfies the conditions for controllability.*

Proof. Cf. Appendix 4. ■

Once controllability of system M can be established, the OGY algorithm requires that the eigenvalues of the controlled system be chosen such that stability is implied. Stabilizing a system is thus translated into searching for values of the nominal, steady state interest rate, \bar{R} , such that all eigenvalues exhibit a negative real part (see Appendix 4).

From Proposition 1, we know that there is a critical value, $|\hat{u}_{cm}^*(\bar{R})|$, such that if $|u_{cm}^*(\bar{R})| > |\hat{u}_{cm}^*(\bar{R})|$, then an active-passive monetary-fiscal regime implies stability of the intended steady state. For notational convenience, let us define

$$|\hat{u}_{cm}^*(\bar{R})| - |u_{cm}^*(\bar{R})| = \theta(\bar{R})$$

and denote

$$\begin{aligned}\bar{R}_\theta^+ &= \{\bar{R} : \theta(\bar{R}) > 0\}, \\ \bar{R}_\theta^- &= \{\bar{R} : \theta(\bar{R}) < 0\}.\end{aligned}$$

We are now ready to prove the following.

specifically, they find that by an inflation-sensitive revenue schedule of the generic form $\tau(a) = \alpha(\pi)a + Rm$ the Government may manage to avert the unintended low-inflation equilibrium. However, this exercise implies a modification of the standard model presented in Section 1. We therefore leave this type of analysis to future research.

³³ The choice of policy instrument or of market intervention operating procedure, to be used in that intermediate targeting, could depend upon the mechanism design of the central bank, which is not a topic of this research. An alternative OGY procedure could use the long run inflation rate, π^* , instead of the nominal interest rate, \bar{R} . Although we believe that the two procedures would likely prove to be mathematically equivalent, we anticipate that an intermediate targeting OGY procedure using \bar{R} would be more easily implemented by a Central Bank than an OGY procedure using the long run inflation rate, which is a final target of policy. From a more technical point of view, and in order to operate an informed choice between π^* and \bar{R} , we have also computed and evaluated at the intended steady-state, the partial derivatives of $G(J)$ in (A.5) with regard to π and R . This computation is helpful since in correspondence of a saddle-focus, the Jacobian of system M presents negative $Tr(J)$ and $Det(J)$ at the intended steady-state. Therefore, chaos control according to the OGY mechanism translates into varying \bar{R} or π^* in such a way that $G(J)$ becomes negative. We found that, for parameters as in Example 3, $\left[\frac{\partial G(J)}{\partial \pi}\right]_{p^*} \ll \left[\frac{\partial G(J)}{\partial R}\right]_{p^*}$, implying that using \bar{R} has proportionally much higher stabilization power than varying π^* .

Proposition 3. (Chaos control in the sticky-price NK model with money in the utility function) Consider the case in which the policy-maker runs an active-passive monetary-fiscal regime and assume that $\bar{R} \in \bar{R}_\theta^+$. Then, by Proposition 1, one eigenvalue is negative, and two eigenvalues have positive real parts. Assume, furthermore, that the economy evolves within a chaotic attractor. Suppose the policy maker announces commitment to a higher steady state nominal interest rate, belonging to the set \bar{R}_θ^- . Then, the economy supersedes irregular and cyclical behavior and approaches the intended steady state.

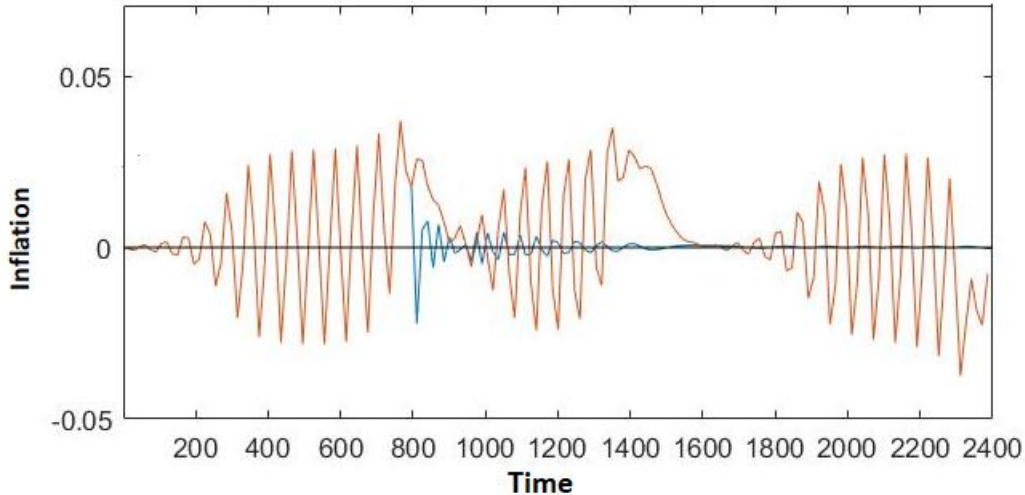
Proof. See Appendix 5. ■

Example 4. Let us denote by $\bar{\mathbf{D}}'$ the difference set $\bar{\mathbf{D}} - \{\beta\}$ and consider the following calibration

$$\bar{\mathbf{D}}' = \hat{\mathbf{D}}' \equiv (350, 0.78966, 21, 1, 0.018, 2).$$

Set the triplet (\bar{R}, π^*, C) at $(0.06, 0.042, 1.5)$, as in the preceding examples, and set $\beta = 1.78$.³⁴ Re-running the algorithm for the presence of a chaotic attractor, we find that system M has a saddle-focus equilibrium with positive saddle quantity and a family of homoclinic orbits for values of the bifurcation parameter τ' belonging to the (extended) interval, $I_{\tau'} \cong (0.02613, 0.27923)$. Set $\tau' = 0.15$. For this specific parameter configuration, J has three eigenvalues with negative real parts for any $\bar{R} \geq 0.06767$. Let us select $\bar{R} = 0.07$ and apply the OGY algorithm (as described in Appendix 4) to obtain the controlled system. In Figure 5, we superimpose the time profile of inflation (red curve), where the control has been initiated at iteration 800.

Figure 5 Un-controlled and controlled inflation rate (control activated at the 800th iteration).



The result in Example 4 might appear puzzling when compared to the policy prescriptions suggested by Benhabib, Schmitt-Grohé, and Uribe (2002). However, a closer look at the different comparative statics

³⁴ Since the two elasticities, β and Φ are very close, there is a very large divide between $|u_{cm}^*(\bar{R})|$ and $|\hat{u}_{cm}^*(\bar{R})|$; and \bar{R} has to undergo a too large jump to make $|u_{cm}^*(\bar{R})| > |\hat{u}_{cm}^*(\bar{R})|$. In this example, we have therefore slightly decreased β .

implied by our model provides a full explanation of this apparent contradiction.³⁵ Recall that Benhabib, Schmitt-Grohè, and Uribe (2002) maintain throughout the paper the complementarity condition in the utility function between real balances and consumption, $u_{cm} > 0$; this implies that a *drop* of the interest rate increases consumption and real economic activity in the model, via the implied increase in money holdings. In our case, the same stimulus to the real activity of the economy is obtained by an *increase* of the interest rate, provided that we have assumed $u_{cm} < 0$ (cf. Assumption 2 above).³⁶ In Benhabib, Schmitt-Grohè, and Uribe (2002), stability is achieved through the simultaneous modulation of nominal interest rates.³⁷ In our case, instead, it is the *commitment* to a long-run easing or tightening of the monetary policy stance, which is able to re-anchor expectations to the long-run target of inflation.

As a check on robustness of our conclusions to our assumption of money in the utility function, we repeat our analysis of OGY chaos control to the alternative specification of money in the production function in Appendix 6.

5. Conclusions

Using the Shilnikov criterion, we find bifurcation to chaos in a NK model at plausible settings of parameters with common NK policy design. The existence of chaos is consistent with the fact that economists who provide short term forecasts are rarely willing to provide long term forecasts. Within the Shilnikov chaos attractor set, we find a downward bias in the interest rate and inflation orbits producing a phenomenon similar to a liquidity trap. The problems associated with the zero-lower bound on nominal interest rates would thereby not be an intentional objective of central bank policy but of the dynamics of the system within the attractor set. The existence of this downward bias, evident for four decades of declining interest rates and inflation rates, has produced the puzzle of very low real rates of interest substantially below the marginal product of capital.

That puzzle has often been observed and frequently unconvincingly imputed to oversaving.³⁸ Our explanation is different and has different policy implications. Paradoxically, an active interest rate feedback policy can cause nominal interest rates, inflation rate, and real interest rates unintentionally to

³⁵ Cf. Wang and Yip (1992), Table 1, p. 555, for a complete derivation of the comparative statics of models with productive and non-productive money.

³⁶ Other crucial differences deserve to be mentioned. Benhabib, Schmitt-Grohè, and Uribe (2002) modify the behavioral rules of public authorities, while the OGY algorithm merely uses a parameter bifurcation approach to induce a change in the topology of the dynamic system. Also our case is aimed at stabilizing irregular chaotic dynamics, whereas Benhabib, Schmitt-Grohè, and Uribe (2002) face the problem of eliminating a liquidity trap caused by an unintended steady state.

³⁷ Benhabib, Schmitt-Grohè, and Uribe (2002) present two examples involving a Taylor rule stipulating *low* (in fact zero) nominal interest rates at low rates of inflation. In one of the examples, the paths leading to the liquidity trap are characterized by nominal interest rates that converge to zero but never actually reach that floor. In the other example, the nominal interest rate hits the zero bound in finite time. Here the central bank must be committed to lower nominal interest rates to zero as inflation becomes sufficiently low.

³⁸ Regarding that popular, but controversial, “loanable funds” explanation, see, e.g., Bofinger and Ries (2017). Paradoxically, that explanation is the opposite of the frequent overconsumption (under saving) explanation of the US balance of payments deficit with China. See, e.g., Hanke (2019) and Hanke and Li (2019).

drift downward, as a result of the dynamical response of the economy within a Shilnikov attractor set. Unlike an open loop interest rate rule, which would directly control an interest rate, active interest rate feedback rules are closed loop rules that link an interest rate to the dynamics of the rest of the economy.

Our results strikingly resemble experience in developed economies in recent decades. We also find and investigate other complications for policy associated with Shilnikov chaos of a NK model.

We propose two potential classes of solutions to the problem:

- (a) The policy design could be altered from an active NK interest rate policy to a fundamentally different design, such as active fiscal policy with passive monetary policy. That alternative is known to produce its own problems in economic dynamics with NK models, but not Shilnikov chaos. Among the many other alternatives in the literature is the targeting of a Divisia monetary aggregate, in accordance with such proposals as those in Belongia (1996), Serletis (2013), or Belongia and Ireland (2014, 2017, 2018) and as advocated by Lucas (2000, p. 270) in measuring welfare loss from inflation.³⁹ By removing the Taylor rule from the NK model, such alternative approaches could prevent chaotic dynamics from occurring. A long literature exists on alternatives to Taylor Rule policies. See, e.g., Cochrane (2011).
- (b) A Taylor rule with interest rate feedback could continue to be used, but with the resulting Shilnikov chaos controlled through the use of a second policy instrument applied in accordance with the policy procedures advocated by engineers in the literature on controlling chaos. We find that the Ott, Grebogi and Yorke (1990) algorithm could be particularly well suited to that objective.

In subsequent research, we plan to explore robustness of our conclusions to different Taylor rules and to alternatives to the Ricardian fiscal policy laws.⁴⁰ But we do not expect fundamental changes in our conclusions, which are systems theory properties of New Keynesian macroeconomic dynamics, when augmented by the closed loop interest rate feedback rules, characterizing all Taylor rules.

References

Afraimovich, V. S., S. V. Gonchenko, L. M. Lerman, A. L. Shilnikov, and D. V. Turaev (2014), “Scientific Heritage of L. P. Shilnikov,” *Regular and Chaotic Dynamics*, 19(4), pp. 435-460.

Algaba, A., Freire, E. Gamer, E., and Rodriguez-Luis, A.J. (1998). “Analysis of Hopf and Takens-

³⁹ Lucas (2000, p. 270) wrote: “I share the widely held opinion that M1 is too narrow an aggregate for this period [the 1990s], and I think that the Divisia approach offers much the best prospects for resolving this difficulty.” Similarly, the International Monetary Fund (2008, pp. 183-184) has advocated Divisia monetary aggregation, available for the U.S. from the Center for Financial Stability (www.centerforfinancialstability.org/amfm.php) and for the UK from the Bank of England. Divisia monetary aggregates are also available at the European Central Bank and many other central banks, but only for internal use.

⁴⁰ Robustness of our conclusions also could be investigated relative to other changes in our standard NK model. In Appendix 6, we have done so relative to replacing money in the utility function with money in the production function. Our conclusions about Shilnikov chaos, when a Taylor rule is adjoined to the model are unchanged. In addition, our conclusion about the relevancy of the OGY algorithm to control the chaos, or alternatively removal of the Taylor rule to end the chaos, remain relevant. But we find that the OGY long run anchor setting to control the chaos is affected. Future research could investigate the general sensitivity of the OGY policy to specification changes in the NK model.

Bogdanov Bifurcations in a Modified van der Pol-Duffing Oscillator.” *Nonlinear Dynamics*, 16, pp. 369-404.

Ascari, G., Florio, A. and Gobbi, A. (2017). “Controlling Inflation with Switching Monetary and Fiscal Policies: Expectations, Fiscal Guidance and Timid Regime Changes.” *Bank of Finland Research Discussion Papers* 9/2017.

Barnett, W. A. And Chen, P. (1988a), “The Aggregation Theoretic Monetary Aggregates Are Chaotic and Have Strange Attractors: an Econometric Application of Mathematical Chaos.” In W. A. Barnett, E. Berndt, and H. White (eds.), *Dynamic Econometric Modeling, Proceedings of the 3rd International Symposium on Economic Theory and Econometrics*, Cambridge U. Press, pp. 199-246. Reprinted in W. A. Barnett and J. M. Binner, *Functional Structure and Approximation in Econometrics*, Elsevier, Amsterdam, 2004, chapter 22.

Barnett, W. A. and Chen, P. (1988b), “Deterministic Chaos and Fractal Attractors as Tools for Nonparametric Dynamical Econometric Inference, *Journal of Mathematical Modeling*, 10(4), pp. 275-296.

Barnett, W. A. and Chen, G. (2015), "Bifurcation of Macroeconometric Models and Robustness of Dynamical Inferences." *Foundations and Trends in Econometrics*, 8(1-2), pp. 1-144.

Barnett, W.A., Duzhak, E. A. (2008), “Non-robust dynamic inferences from macroeconomic models: Bifurcation Stratification of Confidence Regions.” *Physica A*, 387(15), pp. 3817-3825.

Barnett W.A. and Duzhak, E. A. (2010), “Empirical Assessment of Bifurcation Regions within New Keynesian Models,” *Economic Theory*, 45, 99-128.

Barnett, W. A., and Duzhak, E. A. (2019), “Structural Stability of the Generalized Taylor Rule.” *Macroeconomic Dynamics*, 23 (4), June, pp. 1664-1678.

Barnett, W. A. and Ghosh, T. (2013), “Bifurcation Analysis of an Endogenous Growth Model.” *Journal of Economic Asymmetries*, 10, pp. 53-64.

Barnett, W. A. and Ghosh, T. (2014), “Stability Analysis of Uzawa-Lucas Endogenous Growth Model.” *Econ. Theory Bull.*, 2, pp. 33-44.

Barnett, W. A., Gallant, A. R., Hinich, M. J., Jungeilges, J., Kaplan, D., and Jensen, M. J. (1997). “A Single-Blind Controlled Competition between Tests for Nonlinearity and Chaos. *Journal of Econometrics*, 82, pp. 157-192.

Bella G., Mattana P., and Venturi B. (2017), “Shilnikov chaos in the Lucas model of endogenous growth.” *Journal of Economic Theory*, 172, pp. 451-477.

Belongia, M. (1996), “Measurement Matters: Recent Results from Monetary Economics Reexamined,” *Journal of Political Economy* 104, pp. 1065-1083.

- Belongia, M., and Ireland, P. (2014), "The Barnett Critique after Three Decades: A New Keynesian Analysis." *Journal of Econometrics*, 183(1), pp. 5-21.
- Belongia, M., and Ireland, P. (2017), "Circumventing the Zero Lower Bound with Monetary Policy Rules Based on Money." *Journal of Macroeconomics*, 54 (Part A), pp. 42-58.
- Belongia, M., Ireland, P. (2018), "Targeting Constant Money Growth at the Zero Lower Bound." *International Journal of Central Banking*, March, pp. 159-204.
- Benhabib, J., Schmitt-Grohé, S., and Uribe, M. (2001a), "Monetary Policy and Multiple Equilibria." *American Economic Review*, 91(1), pp. 167-186.
- Benhabib, J., Schmitt-Grohé, S., and Uribe, M. (2001b), "The Perils of Taylor Rules." *Journal of Economic Theory*, 96(1-2), pp. 40-69.
- Benhabib, J., Schmitt-Grohé, S., and Uribe, M. (2002), "On Taylor Rules and Monetary Policy: Chaotic Interest-Rate Rules." *AEA Papers and Proceedings*, 92(2), pp. 72-78.
- Bernanke, Ben S. (2020), "The New Tools of Monetary Policy, American Economic Association Presidential Address, January 4, https://www.brookings.edu/wp-content/uploads/2019/12/Bernanke_ASSA_lecture.pdf, with summary on the Brookings Institution web site at https://www.brookings.edu/wp-content/uploads/2019/12/Bernanke_ASSA_lecture.pdf.
- Bhattarai, S., Lee, J. W., and Park, W. Y. (2012), "Monetary-Fiscal Policy Interactions and Indeterminacy in Postwar US Data." *American Economic Review: Papers & Proceedings* 2012, 102(3), pp. 173–178.
- Blanchard, O. J. and Fischer, S. (1989), *Lectures on Macroeconomics*, MIT Press, Cambridge, MA.
- Bofinger, P. and Ries, M. (2017), "Excess Saving and Low Interest Rates: Theory and Empirical Evidence," CEPR Discussion paper 12111.
- Bullard, J. (2010), "Seven Faces of 'The Peril'." *The Federal Reserve Bank of St. Louis Review*, September/October 2010, 92(5), pp. 339-52.
- Carlstrom, C. T. and Fuerst, T. S. (2001), "Timing and real indeterminacy in monetary models." *Journal of Monetary Economics*, 47, 285–298.
- Carlstrom, C. T., and Fuerst, T. S. (2003), "Backward-Looking Interest-Rate Rules, Interest-Rate Smoothing, and Macroeconomic Instability." *Journal of Money, Credit and Banking*, 35(6), pp. 1413-1423.
- Champneys, A. (2010), "To the Memory of L. P. Shilnikov." *The Dynamical Systems Web*, <https://dsweb.siam.org/The-Magazine/All-Issues/to-the-memory-of-lp-shilnikov-1>.

- Chen, B., and Zhou, T. (2011), "Shilnikov homoclinic orbits in two classes of 3D autonomous nonlinear systems." *International Journal of Modern Physics B*, 25(20), pp. 2697-2712.
- Christiano, L and Takahashi, Y. (2018), "Discouraging Deviant Behavior in Monetary Economics," NBER Working Paper 24949, August.
- Cochrane, J. H. (2011), "Determinacy and Identification with Taylor Rules," *Journal of Political Economy*, vol. 119, no. 3, June, pp. 565-615.
- Coibion, O. and Gorodnichenko, Y. (2011), "Monetary Policy, Trend Inflation, and the Great Moderation: An Alternative Interpretation." *American Economic Review*, 101, 341-70.
- Davig, T., and Leeper, E. M. (2011), "Monetary–fiscal policy interactions and fiscal stimulus." *European Economic Review*, 55 (2), pp. 211-227.
- Edge, R. M., and Rudd, J. B. (2007), "Taxation and the Taylor principle." *Journal of Monetary Economics*, 54, pp. 2554-2567.
- Farmer, J.D., Ott E., Yorke, J.A. (1983), "The dimension of chaotic attractors." *Physica D: Nonlinear Phenomena*, 7, pp. 153-180.
- Feenstra, R. C. (1986), "Functional Equivalence between Liquidity Costs and the Utility of Money," *Journal of Monetary Economics* 17, pp. 271-291.
- Freire, E. Gamero, E. Rodriguez-Luis, A.J., Algaba, A. (2002), "A note on the triple-zero linear degeneracy: normal forms, dynamical and bifurcation behaviors of an unfolding." *International Journal of Bifurcation and Chaos*, 12, pp. 2799-2820.
- Galí, J., López- Salido, J. D., and Vallés, J. (2004), "Rule- of- Thumb Consumers and the Design of Interest Rate Rules." *Journal of Money Credit and Banking*, 36, pp. 739-63.
- Geweke, J. (1992), "Inference and Prediction in the Presence of Uncertainty and Determinism." Comment on L. M. Berliner, "Statistics, Probability, and Chaos," and S. Chatterjee and M. Yilmaz, "Chaos, Fractals, and Statistics", *Statistical Science* 7, pp. 94-101.
- Geweke, J. and Durham, G. (2019). "Sequentially Adaptive Bayesian Learning for Inference and Optimization," *Journal of Econometrics* 2010: 4-25.
- Geweke, J., Barnett, W. A., and Shell, K. (1989), *Economic Complexity: Chaos, Sunspots, Bubbles, and Nonlinearity*, Cambridge University Press.
- Grandmont, J. M. (1985), "On endogenous competitive business cycles." *Econometrica*, 53, pp. 995-1045.
- Hanke, S. (2019), "Trade Wars: Facts and Fallacies." *Forbes magazine*, October 15, 3:59 pm.

- Hanke, S. and Li, E. (2019), "The Strange and Futile World of Trade Wars," *Journal of Applied Corporate Finance*, Fall.
- Kaas, L. (1998), "Stabilizing chaos in a dynamic macroeconomic model." *Journal of Economic Behavior & Organization*, 33, pp. 313-332.
- Kiley, M. T. (2007), "Is Moderate-to-High Inflation Inherently Unstable?" *International Journal of Central Banking*, 3(2), pp. 173-201.
- Kumhof, M., Nunes, R., and Yakadina, I. (2010), "Simple monetary rules under fiscal dominance." *J Money Credit Bank*, 42(1), pp. 63-92.
- Kuznetsov, N. V. (2016), "The Lyapunov Dimension and Its Estimation Via the Leonov Method," *Physics Letters* 380(25-26), June, pp. 2142-2149.
- Kuznetsov, Y.A. (1998). *Elements of Applied Bifurcation Theory*. 2nd edition. New York: Springer-Verlag.
- Le Riche, A., Magris, F., and Parent, A. (2017), "Liquidity Trap and stability of Taylor rules." *Mathematical Social Sciences*, 88, pp. 16-27.
- Leeper, E.M. (1991), "Equilibria under 'active' and 'passive' monetary and fiscal policies." *Journal of Monetary Economics*, 27(1), pp. 129-147.
- Lucas, R. E. (2000), "Inflation and welfare." *Econometrica*, 68(62), pp. 247-274.
- Mendes, V. M., and Mendes D. A. (2006), "Active Interest Rate Rules and the Role of Stabilization Policy R&D Tax Credits." Working Papers Series 1, ercwp0208, ISCTE-IUL, Business Research Unit (BRU-IUL).
- Naimzada, A. K., and Sordi, S. (2018), "On controlling chaos in a discrete-time Walrasian tâtonnement process." *Metroeconomica*, 69, pp. 178–194.
- Natvik, G. J. (2009), "Government Spending and the Taylor Principle." *Journal of Money, Credit and Banking*, 41(1), pp. 57-77.
- Ott, E., Grebogi, C, and Yorke, J. A. (1990), "Controlling chaos." *Phys. Rev. Lett.*, 64, 1196-1199.
- Peters, E. E. (1991). *Chaos and Order in the Capital Markets*. John Wiley & Sons, Inc. Publisher.
- Piazza, R. (2016), "Self-fulfilling deflations." *Journal of Economic Dynamics and Control*, 73, pp. 18-40.
- Poterba, J. M. and Rotemberg, J. J. (1987). "Money in the Utility Function." In Barnett, W. A. and Singleton, K.J. (eds.), *New Approaches to Monetary Economics*, Cambridge: Cambridge University Press, pp. 219-240.

- Røisland, Ø. (2003), "Capital income taxation, equilibrium determinacy, and the Taylor principle." *Economics Letters*, 81(2), pp. 147-153.
- Serletis, A., and Rahman, S. (2013), "The Case for Divisia Money Targeting." *Macroeconomic Dynamics*, 17, pp. 1638-1658.
- Serletis, A. and Xu, L. (2019), "Consumption, Leisure, and Money," *Macroeconomic Dynamics*, 23, pp. 1 – 30.
- Shang, D., and Han, M. (2005), "The existence of homoclinic orbits to saddle-focus." *Applied Mathematics and Computation*, 163, pp. 621-631.
- Shilnikov, L.P. (1965), "A case of the existence of a denumerable set of periodic motions." *Sov. Math. Docl.*, 6, pp. 163-166.
- Stokes, H.H. (2016), "Using nonlinear testing procedures to specify the right-hand side of an aggregate production function containing financial variables in the period 1967–2011." *The Journal of Economic Asymmetries*, 14, pp. 147-156.
- Sveen, T., and Weinke, L. (2007), "Firm-specific capital, nominal rigidities, and the Taylor principle." *Journal of Economic Theory*, 136(1), pp. 729-737.
- Sveen, T., and Weinke, L. (2005), "New perspectives on capital, sticky prices, and the Taylor principle." *Journal of Economic Theory*, Volume 123, Pages 21-39.
- Tsuzuki, E. (2016), "Fiscal policy lag and equilibrium determinacy in a continuous-time New Keynesian model." *Int Rev Econ*, 63, 215-232.
- Walsh, C. E. (2010), *Monetary Theory and Policy*, MIT Press, Cambridge, MA, Third Edition.
- Wang, P. and Yip, C. (1992). "Alternative Approaches to Money and Growth," *Journal of Money, Credit, and Banking* 24(4), pp. 553-562.
- Woodford, M. (2003). *Interest and prices: foundations of a theory of monetary policy*. Princeton University Press, Princeton.
- Xu, L. and Serletis, A. (2016), "Monetary and fiscal policy switching with time-varying volatilities." *Economics Letters*, 145, pp. 202-205.

APPENDICES

Appendix 1: Proof of Proposition 1.

Let \mathbf{J} denote the Jacobian matrix of system M , evaluated at the long-run equilibrium, and let starred values denote steady-state state levels. Simple algebra leads to the following (3×3) matrix,

$$J = \begin{bmatrix} 0 & (1 - R'(\pi^*))\mu_1^* & 0 \\ j_{21}^* & j_{22}^* & 0 \\ -\frac{\bar{R}}{u_{mm}^*} \left[\bar{R} - u_{cm}^* \frac{u_{mm}^* - u_{cm}^* \bar{R}}{u_{cc}^* u_{mm}^* - u_{cm}^{*2}} \right] & 0 & \bar{R} - \pi^* - \tau'(a^*) \end{bmatrix}, \quad (\text{A.1})$$

where $j_{21}^* = -\frac{\psi\phi}{\eta} c^* \psi \frac{u_{mm}^* - u_{cm}^* \bar{R}}{u_{cc}^* u_{mm}^* - u_{cm}^{*2}} + \frac{(\phi-1)}{\eta} c^*$, and $j_{22}^* = \rho - \frac{\psi\phi c^* \psi u_{cm}^* \mu_1^*}{\eta(u_{cc}^* u_{mm}^* - u_{cm}^{*2})} R'(\pi^*)$.

By relations in (1) and (4.b), and since $\phi > 1$, we know that $j_{21}^* > 0$. The eigenvalues of \mathbf{J} are the solutions of the characteristic equation

$$\det(\lambda I - J) = \lambda^3 - \text{Tr}(J)\lambda^2 + B(J)\lambda - \text{Det}(J),$$

where \mathbf{I} is the identity matrix and where

$$\text{Tr}(J) = j_{22}^* + \bar{R} - \pi^* - \tau'(a^*), \quad (\text{A.2})$$

$$\text{Det}(J) = [\bar{R} - \pi^* - \tau'(a^*)][R'(\pi^*) - 1]\mu_1^* j_{21}^*, \quad (\text{A.3})$$

$$B(J) = [R'(\pi^*) - 1]\mu_1^* j_{21}^* + [\bar{R} - \pi^* - \tau'(a^*)]j_{22}^* \quad (\text{A.4})$$

are Trace, Determinant, and Sum of principal minors of \mathbf{J} , respectively. Also define

$$G(J) = -B(J) + \frac{\text{Det}(J)}{\text{Tr}(J)}. \quad (\text{A.5})$$

Under the active monetary policy ($R'(\pi^*) > 1$), we now determine the sign of the real parts of the eigenvalues with active or passive fiscal policy. Consider first the case in which the fiscal policy is passive, implying $\bar{R} - \pi^* - \tau'(a^*) < 0$. In this case, $\text{Det}(J) < 0$. Consider instead the case of active fiscal policy, implying $\bar{R} - \pi^* - \tau'(a^*) > 0$. Then, both $\text{Tr}(J)$ and $\text{Det}(J)$ are positive. In this case, irrespective of the sign of $G(J)$, we have one eigenvalue with negative real part and two eigenvalues with positive real parts: P^* is a saddle of index 2, and the equilibrium is locally unique.

We now consider the local stability properties of system M in the neighborhood of P^* , when monetary policy is active. Assume that the monetary policy is active. Then, if the fiscal policy is also active, P^* is a repeller, and there are no equilibrium paths besides the steady-state itself. Conversely, if the fiscal policy is passive, P^* is a saddle of index 2, and the equilibrium is locally unique. ■

Appendix 2: Proof of Proposition 2.

Assume that the conditions in Lemmas 1 and 2 are satisfied. If we start in the neighborhood of the origin, we know from Theorem 1 that, in the phase space of system (22), the solution trajectories are bounded to evolve forever in the neighborhood of the origin and are therefore valid equilibria. By construction, the results obtained for system (21) also apply to the original system of differential equations, M . ■

Appendix 3: Proof of Corollary 1.

Recall that $a(t)$ is the predetermined variable of the system, and that $\mu_1(t)$ and $\pi(t)$ are jump variables. If we choose $a(0)$ to belong to the tubular neighborhood of the homoclinic orbit, as defined above, we know that there must be a *continuum* of possible choices of $\mu_1(0)$ and $\pi(0)$, capable of giving rise to recurrent paths, which are bound to stay forever within E_v . Since all these recurrent paths are bound to stay in a neighborhood of P^* , such that the neighborhood can well exceed the small neighborhood valid for the local analysis, we have in fact global indeterminacy of the equilibrium. ■

Appendix 4: Proof of Lemma 3.

The algorithm for proving controllability of a given system requires that the nonlinear system be written in state-space notation. We first put the linear part of system (20) in the form

$$\dot{w} = Jw + MKw, \quad (\text{A.6})$$

where $w = (w_1, w_2, w_3)^T$, while J is as in A.1. Moreover, $M = \left(\frac{\partial \dot{w}_1}{\partial \bar{R}}, \frac{\partial \dot{w}_2}{\partial \bar{R}}, \frac{\partial \dot{w}_3}{\partial \bar{R}} \right)^T$, while $K = (k_1, k_2, k_3)$ is a (1×3) vector. System (A.6) now needs to be put into its first-companion form,

$$\dot{\omega} = (A - BK)\omega. \quad (\text{A.7})$$

Here $\omega = (\omega_1, \omega_2, \omega_3)^T$ from the following transformation $w = T\omega$, where $A = T^{-1}JT$ is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix}, \quad (\text{A.8})$$

as in (21), and where $B = T^{-1}M$. In detail, the transformation matrix T has to be chosen to satisfy the product $T = NW$, with

$$N = [B, JB, J^2B] \quad (\text{A.9})$$

and

$$W = \begin{bmatrix} \varepsilon_2 & \varepsilon_3 & 1 \\ \varepsilon_3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \quad (\text{A.10})$$

Controllability requires that matrix N have full rank. Since, in our case, matrix A is non-degenerate, the

controllability of system M by means of changes in the nominal rate is feasible. ■

Appendix 5: Proof of Proposition 3.

The OGY algorithm requires that a desired form for the characteristic equation be obtained by varying the control parameter. In our case, the "desired" form implies three eigenvalues with negative real parts. In Proposition 1, we have shown that it can be done, if the policy-maker runs an active-passive monetary-fiscal regime, when $|\hat{u}_{cm}^*(\bar{R})| - |u_{cm}^*(\bar{R})| = \theta(\bar{R}) < 0$. Assume now that the policy maker announces a commitment to a steady state nominal interest rate belonging to the set \bar{R}_θ^- . Then full stability of P^* is affirmed and the statements in the proposition are implied. Since $\frac{\partial u_{cm}^*(\bar{R})}{\partial \bar{R}}$ is invariably positive, there is a commitment to a *higher* steady state nominal interest rate. Example 4 below completes the proof by showing that there are regions of the parameter space such that $\bar{R}_\theta^- \neq \emptyset$. ■

Appendix 6: Chaos and chaos control with money in the production function

Results in Sections 3 and 4 of this paper are based on a NK model with real money balances as an argument in the consumer's utility function. As a check on the robustness of our results to that assumption, we now repeat our analysis with money in the production function, as in Fischer (1974), Stokes (2016), and Benhabib, Schmitt-Grohè and Uribe (2001b). We explore whether the onset of a Shilnikov chaotic attractor is confined to the case of money in the utility or extends to the case of money in the production function. This appendix provides the results of numerical analyses conducted in the NK sticky prices model with money in the production function. We investigate existence of Shilnikov chaos and control of that chaos.

Consider the case in which the utility in (13) is replaced by a single-argument function

$$u(c) = \frac{c^{1-\Phi}-1}{1-\Phi}, \quad (\text{A.11})$$

where c is consumption and where the symbol Φ still denotes the inverse of the elasticity of substitution. Consider, furthermore, introducing real balances into the production function (15) according to the constant-returns Cobb-Douglas formula

$$y(l, m) = Al^\theta m^{1-\theta}, \quad (\text{A.12})$$

where θ is the share of labor so that $(1 - \theta)$ is the share of real balances in production.⁴¹

Given the equations (A.11) and (A.12), we derive the first-order conditions from the Hamiltonian and rearrange the terms, to obtain the following (explicit) system of dynamic laws:

System S:

$$\dot{\mu}_1 = (\rho - R + \pi)\mu_1,$$

⁴¹ Benhabib, Schmitt-Grohè, and Uribe (2001.b) consider a simpler functional form, in which money is the only input in production. Their setting can be retrieved by setting $\theta = 0$.

$$\begin{aligned}\eta\dot{\pi} &= \rho(\pi - \pi^*)\eta - c(\mu_1, \pi) \left[(1 - \phi)\mu_1 + \frac{\phi}{\theta} c(\mu_1, \pi)^{\frac{1+\psi-\theta}{\theta}} m(c(\mu_1, \pi))^{\frac{\theta-1}{\theta}} \right], \\ \dot{a} &= (R - \pi)a - Rm(c(\mu_1, \pi), R) - \tau,\end{aligned}$$

Where, as in system M , the parameter A has been set to one. The calibration of this economy presents several complications. Since we have labor in the production function, we are faced with the calibration of the parameter θ in a constant-returns-to-scale setting. This is problematic since, to the best of our knowledge, that has not previously been done with Cobb-Douglas technology. Therefore, we proceed according to the following steps. We first use θ as a free bifurcation parameter and numerically identify the values at which the chaotic attractor is found. Then, we discuss plausibility of these values with regard to the existing econometric evidence (cf., footnote 28).

Consider the following Example.

Example 5. Denote by \mathbf{F} the set of the deep parameters $(\eta, \phi, \psi, \rho, \Phi)$. Assume

$$\bar{\mathbf{F}} \equiv (350, 21, 1, 0.018, 2) \in \mathbf{F}.$$

and (\bar{R}, π^*, C) at $(0.06, 0.042, 1.5)$, as in the previous examples. Solving the characteristic equation, the first result confirms that, as for the case of the money in utility function model, the complex-conjugate eigenvalues do not depend on the fiscal parameters (τ, τ') . We therefore look for values of the free parameter θ such that the steady-state is a saddle-focus equilibrium. We find that this requires $\theta \in (0.9325, 1)$. Re-running the algorithm in search of a chaotic attractor, we find that the intended steady state in system S satisfies conditions of Theorem 1 for all $\tau' \in I_{\tau'} \cong (0.0534, 0.2754)$. Set therefore $\tau' = 0.15$ as in Example 3. Then, given a triplet of the initial conditions $(w_1(0), w_2(0), w_3(0))$ sufficiently close to the origin, system S admits perfect-foresight chaotic equilibrium solutions.

Therefore, by Example 5, when money is productive, the onset of a chaotic attractor in a sticky-price NK model with a Ricardian fiscal policy cannot be excluded. Comparing the regions of the parameter space, where the intended steady-state is a saddle-focus equilibrium with a positive saddle quantity in systems M and S , we find the following.

Remark 5. Either when money enters the utility function or the production function, the accompanying coefficient must be sufficiently low for the intended long-run equilibrium to be a saddle-focus with a positive saddle quantity. Specifically, given the parameters in the examples above, we have 0.0901 and 0.0675, for money in utility or in production, respectively.⁴²

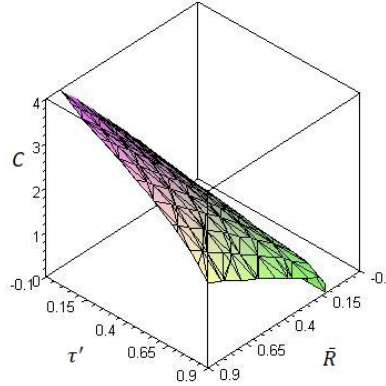
Moreover, to verify the robustness of the results to changes in the parameters, we parallel the analysis of Sub-Section 3.2 and study the surface

⁴² Small coefficients to real balances estimated with Cobb-Douglas production functions are reported in the recent literature (cf., *inter al.* Stokes (2016)). The finding of near constant returns to scale when appropriate measures of labor and capital are used along with exogenous technical change, the study provides a useful benchmark for evaluating the plausibility of low elasticity of real balances.

$$\Omega \equiv \mathbf{B}(\mathbf{J}) + \text{Tr}(\mathbf{J})^2$$

for the money in the production function case. Setting $\mathbf{F} = \bar{\mathbf{F}}$, set $\theta = 0.95$ and using $\pi^* = 0.042$, we obtain Figure 6.

Figure 6. *Combinations of the (C, \bar{R}, τ') parameters at which the intended steady-state of system S is a saddle-focus with $\sigma = 0$.*



The combinations of the remaining (C, \bar{R}, τ') policy parameters, such that the intended steady-state of system S is a saddle-focus equilibrium with no saddle-quantity, are represented. Above the surface, the saddle quantity is positive. Below the surface, the saddle quantity is negative. Figure 6 matches up well with the analogous Figure 1: to have a saddle-focus equilibrium with a positive saddle-quantity, in presence of an active monetary policy ($C > 1$), the pair (\bar{R}, τ') can assume plausibly low values.

We now investigate whether the chaotic motion arising from system S can be OGY-controlled. Recalling that the control parameter must be accessible to the Central Bank, stabilizing chaotic solutions implies the search for long-run values of the policy parameters \bar{R} and π , such that all the eigenvalues of the Jacobian associated with system S have a negative real part.

Consider, therefore, the following numerical example, where we first take \bar{R} as the control parameter.

Example 6. *Set $\mathbf{F} = \bar{\mathbf{F}}$ and $\theta = 0.95$ as in the preceding Example 5. Consider the case of $(\pi^*, C) = (0.042, 1.5)$ and $\tau' = 0.15$. Then, as shown in Example 5, if $\bar{R} = 0.06$, the intended steady state of system S is a saddle-focus equilibrium with a positive saddle quantity. First, we check that system S is controllable. Then, using \bar{R} , as the bifurcation parameter, we see that the Jacobian of S , evaluated at the intended steady-state, has three eigenvalues with negative real parts for any \bar{R} belonging to the interval $I_{\bar{R}} \cong (0.0302, 0.0427)$. Suppose now the policy-maker announces a commitment to $\bar{R} \in I_{\bar{R}}$. Then, by the OGY algorithm, the economy supersedes irregular and cyclical behavior and approaches the intended steady state.*

Notice that, contrary to the money in utility function model, the policy-maker's commitment must be to lower the steady state nominal interest rate.⁴³ Again, as discussed at the end of Section 4, what is

⁴³ The result is invariably confirmed by extensive numerical simulations made on different regions of the “deep” parameters of the economy.

ultimately required for the economy to be stabilized is that the monetary impulse be positive for real economic activity. In the present case of productive money, the comparative statics invariably implies a *positive* partial derivative between c (and therefore y) and money holdings. Therefore, a commitment to a reduction of the long-run interest rate immediately raises money balances.